I. V. Proskuryakov

Problems in Linear Algebra

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by
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Preface

In preparing this book of problems the author attempted, firstly, to give a sufficient number of exercises for developing skills in the solution of typical problems (for example, the computing of determinants with numerical elements, the solution of systems of linear equations with numerical coefficients, and the like), secondly, to provide problems that will help to clarify basic concepts and their interrelations (for example, the connection between the properties of matrices and those of quadratic forms, on the one hand, and those of linear transformations, on the other), thirdly, to provide for a set of problems that might supplement the course of lectures and help to expand the mathematical horizon of the student (instances are the properties of the Pfaffian of the skew-symmetric determinant, the properties of associated matrices, and so on).

A number of problems involving the proof of theorems that can be found in textbooks are also given. These problems were included because the instructor often (for lack of time) gives part of the material as homework for the student on the basis of the textbook, and this can be done on the basis of the problem book where hints are given for working out proofs. The author feels this can help to develop habits of scientific investigation.

Compared with other problem books, this one has a few new basic features. They include problems dealing with polynomial matrices (Sec. 13), linear transformations of affine and metric spaces (Secs. 18 and 19), and a supplement devoted to groups, rings, and fields. The problems of the supplement deal with the most elementary portions of the theory. Still and all, I think it can be used in pre-seminar discussions in the first and second years of study.

The contents and the sequence of presentation of material in a lecture course depend largely on the lecturer. The author has tried to take into account this diversity of pre-

sentation, and the result has been a certain amount of duplication. For example, the same facts are given first in the section devoted to quadratic forms and then again in the chapter on linear transformations; some of the problems are stated so that they can be worked out in the case of a real Euclidean space and also in that of a complex unitary space. I believe that this makes for a certain flexibility of use of the problem book.

Some sections contain an introduction with a few definitions and a brief discussion of terminology and notation when the available textbooks do not exhibit complete unity in this respect. An exception is the introduction to Sec. 5, where basic methods presented for computing determinants of any order and examples of each method are given. This was done because such information is usually not given in standard textbooks, and students encounter considerable difficulties.

Starred numbers indicate problems that have been worked out or provided with hints. Solutions are given for a small number of problems. These are either problems involving a general method that is then applied to a series of other problems (for instance, problem 1151 which offers a method for computing the function of a matrix, problem 1529 which contains the construction of a basis in which the matrix of a linear transformation has Jordan form) or problems that are very difficult (say, problems 1433, 1614, 1617). As a rule, the hints contain only a suggestion of the idea or method of solution and leave to the student the actual solving. Only in the case of more difficult problems is a general plan of solution provided (see problems 546, 1492, 1632).

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I. Proskuryakov

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Chapter I

Determinants

Sec. 1. Second- and Third-Order Determinants

Compute the following determinants:

1. $\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}$. **2.** $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$. **3.** $\begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix}$. **4.** $\begin{vmatrix} 6 & 9 \\ 8 & 12 \end{vmatrix}$.

5. $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$. **6.** $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$. **7.** $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$.

8. $\begin{vmatrix} a^2+ab+b^2 & a^2-ab+b^2 \\ a+b & a-b \end{vmatrix}$. **9.** $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$.

10. $\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$. **11.** $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$.

12. $\begin{vmatrix} \sin\alpha+\sin\beta & \cos\beta+\cos\alpha \\ \cos\beta-\cos\alpha & \sin\alpha-\sin\beta \end{vmatrix}$.

13. $\begin{vmatrix} 2\sin\varphi\cos\varphi & 2\sin^2\varphi-1 \\ 2\cos^2\varphi-1 & 2\sin\varphi\cos\varphi \end{vmatrix}$.

14. $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix}$. **15.** $\begin{vmatrix} \frac{(1-t)^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{2t}{1+t^2} & -\frac{(1+t)^2}{1+t^2} \end{vmatrix}$.

16. $\begin{vmatrix} \frac{1+t^2}{1-t^2} & \frac{2t}{1-t^2} \\ \frac{2t}{1-t^2} & \frac{1+t^2}{1-t^2} \end{vmatrix}$. **17.** $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$.

Compute the following determinants ($i = \sqrt{-1}$):

18. $\begin{vmatrix} a & c+di \\ c-di & b \end{vmatrix}$. **19.** $\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$.

20. $\begin{vmatrix} \cos\alpha + i\sin\alpha & 1 \\ 1 & \cos\alpha - i\sin\alpha \end{vmatrix}$. 21. $\begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix}$.

Use determinants to solve the following systems of equations:

22. $2x+5y=1$, $3x+7y=2$.

23. $2x-3y=4$, $4x-5y=10$.

24. $5x-7y=1$, $x-2y=0$.

25. $4x+7y+13=0$, $5x+8y+14=0$.

26. $x\cos\alpha - y\sin\alpha = \cos\beta$, $x\sin\alpha + y\cos\alpha = \sin\beta$.

27. $x\tan\alpha + y = \sin(\alpha+\beta)$, $x - y\tan\alpha = \cos(\alpha+\beta)$,

where $\alpha \neq \frac{\pi}{2} + k\pi$ (k an integer).

Investigate to see whether the given system of equations is evendetermined (has a unique solution), indeterminate (has an infinity of solutions) or is inconsistent (no solution):

28. $4x+6y=2$, $6x+9y=3$. (Do Cramer's formulas yield a correct answer?)

29. $3x-2y=2$, $6x-4y=3$.

30. $(a-b)x = b-c$.

31. $x\sin\alpha = 1+\sin\alpha$.

32. $x\sin\alpha = 1+\cos\alpha$.

33. $x\sin(\alpha+\beta) = \sin\alpha + \sin\beta$.

34. $a^2x = ab$, $abx = b^2$.

35. $ax+by=ad$, $bx+cy=bd$.

36. $ax+4y=2$, $9y+ay=3$.

37. $ax-9y=6$, $10x-by=10$.

38. Prove that for a determinant of the second order to be equal to zero it is necessary and sufficient that the rows be proportional. The same holds true for columns as well (if certain elements of the determinant are zero, the proportionality may be understood in the sense that the elements of one row are obtained from the corresponding elements of another row by multiplying by the same number, which may even be zero).

*39. Prove that when a, b, c are real, the roots of the equation $\begin{vmatrix} a-x & b \\ b & c-x \end{vmatrix} = 0$ are real.

*40. Prove that the quadratic trinomial $ax^2 + 2bx + c$ with complex coefficients is a perfect square if and only if

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = 0$$

41. Prove that for real a, b, c, d, the roots of the equation $\begin{vmatrix} a-x & c+di \\ c-di & b-x \end{vmatrix} = 0$ are real.

*42. Show that the value of the fraction $\frac{ax+b}{cx+d}$, where at least one of the numbers c, d is nonzero, is not dependent on the value of x if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$.

Compute the following third-order determinants:

43. $\begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix}$. 44. $\begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 2 \end{vmatrix}$. 45. $\begin{vmatrix} 4 & -3 & 5 \\ 3 & -2 & 8 \\ 1 & -7 & -5 \end{vmatrix}$.

46. $\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$. 47. $\begin{vmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}$. 48. $\begin{vmatrix} 4 & 2 & -1 \\ 5 & 3 & -2 \\ 3 & 2 & -1 \end{vmatrix}$.

49. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$. 50. $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$. 51. $\begin{vmatrix} 5 & 6 & 3 \\ 0 & 1 & 0 \\ 7 & 4 & 5 \end{vmatrix}$.

52. $\begin{vmatrix} 2 & 0 & 3 \\ 7 & 1 & 6 \\ 6 & 0 & 5 \end{vmatrix}$. 53. $\begin{vmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{vmatrix}$. 54. $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 9 \\ 16 & 25 & 81 \end{vmatrix}$.

55. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. 56. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. 57. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

58. $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$. 59. $\begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix}$. 60. $\begin{vmatrix} a+x & x & x \\ x & b+x & x \\ x & x & c+x \end{vmatrix}$.

61. $\begin{vmatrix} \alpha^2+1 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2+1 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2+1 \end{vmatrix}$.

62. $\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{vmatrix}$.

63. $\begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix}$.

64. Determine under what condition the following equation holds true:

$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}.$$

65. Show that the determinant

$$\begin{vmatrix} a^2 & b\sin\alpha & c\sin\alpha \\ b\sin\alpha & 1 & \cos\alpha \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix}$$

and two other determinants obtained from this one by a circular permutation of the elements a, b, c and α, β, γ are equal to zero if a, b, c are the lengths of the sides of a triangle and α, β, γ are the angles opposite the sides a, b, c, respectively.

Evaluate the following third-order determinants ($i = \sqrt{-1}$):

66. $\begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix}$.

67. $\begin{vmatrix} x & a+bi & c+di \\ a-bi & y & e+fi \\ c-di & e-fi & z \end{vmatrix}$.

68. $\begin{vmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{vmatrix}$, where $\varepsilon = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$.

69. $\begin{vmatrix} 1 & 1 & \varepsilon \\ 1 & 1 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon & 1 \end{vmatrix}$, where $\varepsilon = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$.

70. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{vmatrix}$, where $\varepsilon = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi$.

71. Prove that if all the elements of a third-order determinant are equal to ± 1, then the determinant itself will be an even number.

*72. Find the maximum value that can be assumed by a third-order determinant provided that all its elements are equal to ± 1.

*73. Find the largest value of a third-order determinant provided that the elements are equal to $+1$ or 0.

Use determinants to solve the following systems of equations:

74. $\begin{aligned} 2x+3y+5z &= 10, \\ 3x+7y+4z &= 3, \\ x+2y+2z &= 3. \end{aligned}$

75. $\begin{aligned} 5x-6y+4z &= 3, \\ 3x-3y+2z &= 1, \\ 4x-5y+2z &= 1. \end{aligned}$

76. $\begin{aligned} 4x-3y+2z+4 &= 0, \\ 6x-2y+3z+1 &= 0, \\ 5x-3y+2z+3 &= 0. \end{aligned}$

77. $\begin{aligned} 5x+2y+3z+2 &= 0, \\ 2x-2y+5z &= 0, \\ 3x+4y+2z+10 &= 0. \end{aligned}$

*78. $\begin{aligned} \frac{x}{a}-\frac{y}{b}+2 &= 0, \\ -\frac{2y}{b}+\frac{3z}{c}-1 &= 0, \\ \frac{x}{a}+\frac{z}{c} &= 0. \end{aligned}$

79. $\begin{aligned} 2ax-3by+cz &= 0, \\ 3ax-6by+5cz &= 2abc, \\ 5ax-4by+2cz &= 3abc, \end{aligned}$
where $abc \neq 0$.

*80. $\begin{aligned} 4bcx+acy-2abz &= 0, \\ 5bcx+3acy-4abz+abc &= 0, \\ 3bcx+2acy-abz-4abc &= 0 \quad (abc \neq 0). \end{aligned}$

*81. Solve the following system of equations:

$\begin{aligned} x+y+z &= a, \\ x+\varepsilon y+\varepsilon^2 z &= b, \\ x+\varepsilon^2 y+\varepsilon z &= c, \end{aligned}$ (ε is a value of $\sqrt[3]{1}$ different from 1).

Investigate each system of equations to determine whether it is evendetermined, indeterminate, or inconsistent:

82. $\begin{aligned} 2x-3y+z&=2,\\ 3x-5y+5z&=3,\\ 5x-8y+6z&=5. \end{aligned}$

83. $\begin{aligned} 4x+3y+2z&=1,\\ x+3y+5z&=1,\\ 3x+6y+9z&=2. \end{aligned}$

84. $\begin{aligned} 5x-6y+z&=4,\\ 3x-5y-2z&=3,\\ 2x-y+3z&=5. \end{aligned}$

85. $\begin{aligned} 2x-y+3z&=4,\\ 3x-2y+2z&=3,\\ 5x-4y&=2. \end{aligned}$

86. $\begin{aligned} 2ax-23y+29z&=4,\\ 7x+ay+4z&=7,\\ 5x+2y+az&=5. \end{aligned}$

87. $\begin{aligned} ax-3y+5z&=4,\\ x-ay+3z&=2,\\ 9x-7y+8az&=0. \end{aligned}$

88. $\begin{aligned} ax+4y+z&=0,\\ 2y+3z-1&=0,\\ 3x-bz+2&=0. \end{aligned}$

89. $\begin{aligned} ax+2z&=2,\\ 5x+2y&=1,\\ x-2y+bz&=3. \end{aligned}$

Prove, either by direct computation via the triangle rule or by the rule of Sarrus, the following properties of third-order determinants:

90. A determinant of the third order remains unchanged if the rows and columns are interchanged (that is to say, if the matrix of the determinant is transposed).

91. If all elements of some row (or column) are equal to zero, then the determinant itself is equal to zero.

92. If all elements of some row (or column) of a determinant are multiplied by one and the same number, the whole determinant is then multiplied by that number.

93. A determinant changes sign if two rows (or two columns) are interchanged.

94. If two rows (or two columns) of a determinant are the same, the determinant is zero.

95. If all elements of one row are proportional to the corresponding elements of another row, the determinant is equal to zero (the same holds true for the columns).

96. If each element of some row of a determinant is represented as the sum of two terms, then the determinant is equal to the sum of two determinants in which all rows, except the given row, remain the same, while the given row in the first determinant contains the first terms and the given row in the second determinant contains the second terms (the same holds true for the columns).

97. A determinant remains unchanged if elements of one row of the determinant are added to the corresponding elements of another row multiplied by the same number (the same applies to columns).

98. We say that one row of a determinant is a *linear combination* of the other rows if each element of the given row is equal to the sum of the products of the corresponding elements of the other rows into certain numbers that are constant for each row, that is to say, that are independent of the position number of the element in the row. A similar definition applies to a linear combination of columns. For example, the third row of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is a linear combination of the first two if there exist numbers c_1 and c_2 such that $a_{3,j} = c_1 a_{1,j} + c_2 a_{2,j}$ $(j = 1, 2, 3)$.

Prove that if one row (or column) of a third-order determinant is a linear combination of the other rows (or columns), the determinant is equal to zero.

Hint. The converse is also true, but it follows from a further development of the theory of determinants.

***99.** Use the preceding problem to demonstrate with an illustration that, unlike second-order determinants (see problem 38), the proportionality of two rows (or columns) is no longer necessary for a third-order determinant to be equal to zero.

Using the properties of third-order determinants given in problems 91-98, compute the following determinants:

100. $\begin{vmatrix} \sin^2\alpha & 1 & \cos^2\alpha \\ \sin^2\beta & 1 & \cos^2\beta \\ \sin^2\gamma & 1 & \cos^2\gamma \end{vmatrix}$.

101. $\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$.

102. $\begin{vmatrix} x & x' & ax+bx' \\ y & y' & ay+by' \\ z & z' & az+bz' \end{vmatrix}$.

103. $\begin{vmatrix} (a_1+b_1)^2 & a_1^2+b_1^2 & a_1b_1 \\ (a_2+b_2)^2 & a_2^2+b_2^2 & a_2b_2 \\ (a_3+b_3)^2 & a_3^2+b_3^2 & a_3b_3 \end{vmatrix}$.

104. $\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix}$.

105. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$.

106. $\begin{vmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon & \varepsilon^2 & 1 \\ x & y & z \end{vmatrix}$, where ε is a value of $\sqrt[3]{1}$ different from 1.

107. $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

108. $\begin{vmatrix} a_1+b_1 i & a_1 i-b_1 & c_1 \\ a_2+b_2 i & a_2 i-b_2 & c_2 \\ a_3+b_3 i & a_3 i-b_3 & c_3 \end{vmatrix}$, where $i=\sqrt{-1}$.

109. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \frac{x_1+\lambda x_2}{1+\lambda} & \frac{y_1+\lambda y_2}{1+\lambda} & 1 \end{vmatrix}$ (give a geometrical interpretation of the result obtained).

*110. $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$, where α, β, γ are roots of the equation $x^3+px+q=0$.

Prove the following identities without expanding the determinants:

111. $\begin{vmatrix} a_1 & b_1 & a_1x+b_1y+c_1 \\ a_2 & b_2 & a_2x+b_2y+c_2 \\ a_3 & b_3 & a_3x+b_3y+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

112. $\begin{vmatrix} a_1+b_1x & a_1-b_1x & c_1 \\ a_2+b_2x & a_2-b_2x & c_2 \\ a_3+b_3x & a_3-b_3x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

113. $\begin{vmatrix} a_1+b_1 i & a_1 i+b_1 & c_1 \\ a_2+b_2 i & a_2 i+b_2 & c_2 \\ a_3+b_3 i & a_3 i+b_3 & c_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

114. $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

115. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b)$.

116. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$

117. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a+b+c)(b-a)(c-a)(c-b).$

118. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (ab+ac+bc)(b-a)(c-a)(c-b).$

119. $\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a^2+b^2+c^2+ab+ac+bc)(b-a)(c-a) \times (c-b).$

***120.** $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

121. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

122. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

Sec. 2. Permutations and Substitutions

Determine the number of inversions in the following permutations (unless otherwise stated, the initial arrangement is taken to be 1, 2, 3, . . . in increasing order):

123. 2, 3, 5, 4, 1. **124.** 6, 3, 1, 2, 5, 4.

125. 1, 9, 6, 3, 2, 5, 4, 7, 8. **126.** 7, 5, 6, 4, 1, 3, 2.

127. 1, 3, 5, 7, . . ., $2n-1$, 2, 4, 6, 8, . . ., $2n$.

128. 2, 4, 6, . . ., $2n$, 1, 3, 5, . . ., $2n-1$.

In the following permutations, determine the number of inversions and indicate the general criterion of those numbers n for which the permutation is even and those for which

it is odd:

129. 1, 4, 7, . . ., $3n - 2$, 2, 5, 8, . . ., $3n - 1$, 3, 6, 9, . . ., $3n$.

130. 3, 6, 9, . . ., $3n$, 2, 5, 8, . . ., $3n - 1$, 1, 4, 7, ..., $3n - 2$.

131. 2, 5, 8, . . ., $3n - 1$, 3, 6, 9, . . ., $3n$, 1, 4, 7, ..., $3n - 2$.

132. 2, 5, 8, . . ., $3n - 1$, 1, 4, 7, . . ., $3n - 2$, 3, 6, 9, . . ., $3n$.

133. 1, 5, . . ., $4n - 3$, 2, 6, . . ., $4n - 2$, 3, 7, ..., $4n - 1$, 4, 8, . . ., $4n$.

134. 1, 5, . . ., $4n - 3$, 3, 7, . . ., $4n - 1$, 2, 6, ..., $4n - 2$, 4, 8, . . ., $4n$.

135. $4n$, $4n - 4$, . . ., 8, 4, $4n - 1$, $4n - 5$, . . ., 7, 3, $4n - 2$, $4n - 6$, . . ., 6, 2, $4n - 3$, $4n - 7$, . . ., 5, 1.

136. In what permutation of the numbers 1, 2, . . ., n is the number of inversions greatest and what is it?

137. How many inversions does the number 1 in the kth position of a permutation generate?

138. How many inversions are generated by the number n in the kth position in a permutation of the numbers 1, 2, 3, . . ., n?

139. What is the sum of the number of inversions and the number of order arrangements in any permutation of the numbers 1, 2, . . ., n?

140. For what numbers n is the parity of the number of inversions and the number of arrangements in all permutations of the numbers 1, 2, . . ., n the same and for what numbers is it different?

***141.** Prove that the number of inversions in the permutation $a_1, a_2, \ldots, a_n$ is equal to the number of inversions in the permutation of the indices 1, 2, . . ., n obtained if the given permutation is replaced by the original arrangement.

***142.** Show that it is possible to pass from one permutation $a_1, a_2, \ldots, a_n$ to another permutation $b_1, b_2, \ldots, b_n$ of the same elements via at most $n - 1$ transpositions.

***143.** Give an illustration of a permutation of the numbers 1, 2, 3, . . ., n which cannot be brought to a normal arrangement in less than $n - 1$ transpositions, and prove this.

*144. Prove that it is possible to pass from one permutation $a_1, a_2, \ldots, a_n$ to any other permutation $b_1, b_2, \ldots, b_n$ of the same elements via at most $\frac{n(n-1)}{2}$ adjacent transpositions (transpositions of adjacent elements, that is).

*145. Given: the number of inversions in the permutation $a_1, a_2, \ldots, a_{n-1}, a_n$ is equal to k. How many inversions are there in the permutation $a_n, a_{n-1}, \ldots, a_2, a_1$?

*146. How many inversions are there in all permutations of n elements taken together?

*147. Prove that it is possible to pass from any permutation of the numbers $1, 2, \ldots, n$, containing k inversions, to the original arrangement by means of k adjacent transpositions, but it is impossible to do so via a smaller number of such transpositions.

*148. Prove that for any integer k $(0 \leqslant k \leqslant C_n^2)$ there exists a permutation of the numbers $1, 2, 3, \ldots, n$, the number of inversions of which is equal to k.

*149. Denote by (n, k) the number of permutations of the numbers $1, 2, \ldots, n$, each of which contains exactly k inversions. For the number (n, k) derive the recurrence relation

$$(n+1, k) = (n, k) + (n, k-1) + (n, k-2) + \ldots + (n, k-n),$$

where (n, j) must be 0 for $j > C_n^2$ and for $j < 0$. Use this relation to set up an array of the numbers (n, k) for $n = 1, 2, 3, 4, 5, 6$ and $k = 0, 1, 2, \ldots, 15$.

*150. Prove that the number of permutations of the numbers $1, 2, \ldots, n$ containing k inversions is equal to the number of permutations of the same numbers containing $C_n^2 - k$ inversions.

Expand the following substitutions into products of independent cycles and determine their parity by the decrement (that is, the difference between the number of actually permuted elements and the number of cycles). To simplify counting the decrement, one-term cycles can be introduced into the expansion for numbers that remain in place.

151. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$.

152. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 4 & 2 & 3 \end{pmatrix}$.

153. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 3 & 6 & 5 & 7 & 4 & 2 \end{pmatrix}$.

154. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 9 & 2 & 1 & 4 & 3 & 6 & 7 \end{pmatrix}$.

155. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{pmatrix}$.

156. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$.

157. $\begin{pmatrix} 1, & 2, & 3, & 4, & \ldots, & 2n-1, & 2n \\ 2, & 1, & 4, & 3, & \ldots, & 2n, & 2n-1 \end{pmatrix}$.

158. $\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & \ldots, & 3n-2, & 3n-1, & 3n \\ 3, & 2, & 1, & 6, & 5, & 4, & \ldots, & 3n, & 3n-1, & 3n-2 \end{pmatrix}$.

159. $\begin{pmatrix} 1, & 2, & 3, & 4, & \ldots, & 2n-3, & 2n-2, & 2n-1, & 2n \\ 3, & 4, & 5, & 6, & \ldots, & 2n-1, & 2n, & 1, & 2 \end{pmatrix}$.

160. $\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & \ldots, & 3n-2, & 3n-1, & 3n \\ 2, & 3, & 1, & 5, & 6, & 4, & \ldots, & 3n-1, & 3n, & 3n-2 \end{pmatrix}$.

161. $\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & \ldots, & 3n-2, & 3n-1, & 3n \\ 4, & 5, & 6, & 7, & 8, & 9, & \ldots, & 1, & 2, & 3 \end{pmatrix}$.

162. $\begin{pmatrix} 1, & 2, & \ldots, & k, & \ldots, & nk-k+1, & nk-k+2, & \ldots, & nk \\ k+1, & k+2, & \ldots, & 2k, & \ldots, & 1, & 2, & \ldots, & k \end{pmatrix}$.

In the following permutations, change from cyclic notation to two-row notation:

163. (1 5) (2 3 4). **164.** (1 3) (2 5) (4).

165. (7 5 3 1) (2 4 6) (8) (9).

166. (1 2) (3 4) . . . ($2n$ — 1, $2n$).

167. (1 2 3 4 . . . $2n$ — 1, $2n$).

168. (3 2 1) (6 5 4) . . . ($3n$, $3n$ — 1, $3n$ — 2).

Multiply the following substitutions:

169. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$.

170. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$.

171. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$.

172. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}^2$.

173. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}^3$.

174. Prove that if the power of a cycle is unity, then the exponent of the power is divisible by the cycle length. (The length of a cycle is the number of its elements.)

175. Prove that from among all powers of a substitution equal to unity the smallest exponent is equal to the smallest common multiple of the lengths of cycles that enter into the decomposition of the substitution.

***176.** Find A^{100}, where $A = \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ 3, 5, 4, 1, 7, 10, 2, 6, 9, 8 \end{pmatrix}$.

177. Find A^{150}, where $A = \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ 3, 5, 4, 6, 9, 7, 1, 10, 8, 2 \end{pmatrix}$.

178. Find the substitution X from the equation $AXB = C$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 7 & 4 & 5 & 6 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 6 & 4 & 7 & 2 \end{pmatrix}.$$

179. Prove that premultiplication of a substitution by a transposition (that is, a two-member cycle) (α, β) is equivalent to a transposition (an interchange) of the numbers α and β in the upper row of the substitution, and postmultiplication by the same transposition is equivalent to a transposition of α and β in the lower row of the substitution.

180. Prove that if the numbers α and β enter into one cycle of a substitution, then in the case of pre- or postmultiplication of the substitution by the transposition (α, β), the given cycle decomposes into two cycles, but if the numbers α and β appear in different cycles, then these cycles merge into one under the given multiplication.

***181.** Use the two preceding problems to prove that the number of inversions and the decrement of any substitution have the same parity.

***182.** Prove that the smallest number of transpositions into the product of which the given substitution is decomposed is equal to its decrement.

***183.** Prove that the smallest number of transpositions that carry the permutation $a_1, a_2, \ldots, a_n$ into the permuta-

tion $b_1, b_2, \ldots, b_n$ of the same elements is equal to the decrement of the substitution

$$P = \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{pmatrix}.$$

*184. Find all substitutions of the numbers 1, 2, 3, 4 that are permutable with the substitution

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

185. Find all substitutions of the numbers 1, 2, 3, 4, 5 that are permutable with the substitution

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}.$$

*186. For any two integers x and m, where $m \neq 0$, denote by $r(x, m)$ the remainder (assumed to be nonnegative) obtained from the division of x by m. Prove that if $m \geqslant 2$ and a is an integer prime to m, then the correspondence $x \to r(ax, m)$, $x = 1, 2, \ldots, m - 1$, is a substitution of the numbers $1, 2, \ldots, m - 1$.

187. Write the substitution of the numbers 1, 2, 3, 4, 5, 6, 7, 8 under which the number x passes into the remainder obtained from the division of $5x$ by 9.

Sec. 3. Definition and Elementary Properties of Determinants of any Order

The problems of this section are aimed at elucidating the concept of a determinant of any order and its most elementary properties, including that of a determinant being equal to zero when the rows are linearly dependent, and the expansion of a determinant in terms of a row.

Problems intended to develop skill in computing determinants with numerical elements, problems involving methods of computing determinants of a special type or dealing with the Laplace theorem, and problems dealing with the multiplication of determinants are all dealt with in the sections that follow Sec. 3.

Determine which of the products given below enter into determinants of appropriate orders and with what signs;

188. $a_{43}a_{21}a_{35}a_{12}a_{54}$. 189. $a_{61}a_{23}a_{45}a_{36}a_{12}a_{54}$.

190. $a_{27}a_{36}a_{51}a_{74}a_{25}a_{43}a_{62}$. 191. $a_{33}a_{16}a_{72}a_{27}a_{55}a_{61}a_{44}$.

192. $a_{12}a_{23}a_{34} \ldots a_{n-1,\,n}a_{kk}$, $1 \leqslant k \leqslant n$.

193. $a_{12}a_{23} \ldots a_{n-1,\,n}a_{n1}$.

194. $a_{12}a_{21}a_{34}a_{43} \ldots a_{2n-1,\,2n}a_{2n,\,2n-1}$.

195. $a_{11}a_{2,\,n}a_{3,\,n-1} \ldots a_{n,\,2}$.

196. $a_{13}a_{22}a_{31}a_{46}a_{55}a_{64} \ldots a_{3n-2,\,3n}a_{3n-1,\,3n-1}a_{3n,\,3n-2}$.

197. Choose the values of i and k so that the product

$$a_{62}a_{i5}a_{33}a_{k4}a_{46}a_{21}$$

appears in a sixth-order determinant with the minus sign.

198. Choose the values of i and k so that the product

$$a_{47}a_{63}a_{1i}a_{55}a_{7k}a_{24}a_{31}$$

enters into a seventh-order determinant with the plus sign.

199. Find terms of a fourth-order determinant containing the element a_{32} and appearing in the determinant with the plus sign.

200. Find terms of the determinant

$$\begin{vmatrix} 5x & 1 & 2 & 3 \\ x & x & 1 & 2 \\ 1 & 2 & x & 3 \\ x & 1 & 2 & 2x \end{vmatrix}$$

that contain x^4 and x^3.

201. What is the sign of a determinant of order n in a product of elements of the principal diagonal?

202. What sign does a determinant of order n have in a product of elements of the secondary diagonal?

203. Using only the definition of a determinant, compute

$$\begin{vmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ a_{31} & a_{32} & a_{33} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix},$$

where all elements above or below the principal diagonal are zero.

204. Using only the definition of a determinant, compute

$$\begin{vmatrix} 0 & \dots & 0 & 0 & a_{1n} \\ 0 & \dots & 0 & a_{2,\,n-1} & a_{2n} \\ 0 & \dots & a_{3,\,n-2} & a_{3,\,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,\,n-2} & a_{n,\,n-1} & a_{nn} \end{vmatrix},$$

where all elements above or below the secondary diagonal are zero.

205. Using only the definition of a determinant, compute

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix}.$$

206. Prove that if a determinant of order n has zero elements at the intersection of certain k rows and l columns, $k + l > n$, the determinant is equal to zero.

***207.** Solve the equation

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ 1 & a_2 & a_2^2 & \dots & a_2^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{vmatrix} = 0,$$

where $a_1, a_2, \dots, a_n$ are distinct numbers.

208.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1-x & 1 & \dots & 1 \\ 1 & 1 & 2-x & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & n-x \end{vmatrix} = 0.$$

209. Find an element of a determinant of order n that is symmetric to the element a_{ik} about the secondary diagonal.

210. Find an element of a determinant of order n that is symmetric to the element a_{ik} about the "centre" of the determinant.

211. We say the position of an element a_{ik} of a determinant is even or odd according as the sum $i + k$ is even or odd. Find the number of elements of a determinant of order n residing in even and odd positions.

212. How will an nth-order determinant change if the first column is put in the last place and the other columns are moved leftwards with all positions retained?

213. What change does a determinant of order n undergo if the rows are given in reversed order?

214. How does a determinant change if each element is replaced by an element symmetric to the given one about the "centre" of the determinant?

***215.** How does a determinant change if each element is replaced by an element symmetric to the given one about the secondary diagonal?

***216.** A determinant is said to be *skew-symmetric* if the elements symmetric about the principal diagonal differ in sign, that is, $a_{ik} = -a_{ki}$ for any indices i, k.

Prove that a skew-symmetric determinant of odd order n is equal to zero.

***217.** Prove that a determinant whose elements are symmetric about the principal diagonal and are complex conjugate numbers (real numbers, as a special case) is a real number.

218. For what values of n will all determinants of order n whose elements satisfy the conditions

$$(\alpha)\ a_{jk} \text{ is real for } j > k,$$

$$(\beta)\ a_{kj} = ia_{jk} \text{ for } j \geqslant k \quad (i = \sqrt{-1})$$

be real?

219. For what values of n will all determinants of order n whose elements satisfy the conditions (α) and (β) of the preceding problem be pure imaginaries?

220. Show that for odd n all determinants of order n whose elements satisfy the conditions (α) and (β) of problem 218 have the form $a(1 \pm i)$ where a is a real number.

221. How will a determinant of order n change if the signs of all its elements are reversed?

***222.** How will a determinant change if each element a_{ik} is multiplied by c^{i-k}, where $c \neq 0$?

***223.** Prove that every term of a determinant involves an even number of elements occupying odd sites; and there is an even number of elements occupying even sites if the deter-

minant is of even order; the number is odd if the determinant is of odd order.

*224. Prove that a determinant remains unchanged if the signs of all elements in odd positions are changed; but if the signs of all elements in even positions are changed, then the determinant remains unchanged if it is of even order, but changes sign if it is of odd order.

225. Prove that a determinant remains unchanged if to each row, with the exception of the last row, the following row is added.

226. Prove that a determinant remains unchanged if to each column (from the second onward) one adds the preceding column.

227. Prove that a determinant remains unchanged if from each row (except the last one) we subtract all succeeding rows.

228. Prove that a determinant remains unchanged if to each column (beginning with the second one) we add all preceding columns.

229. How does a determinant change if from each row (except the last one) we subtract the following row and from the last row we subtract the earlier first row?

*230. What change does a determinant undergo if to each column (beginning with the second) we add the preceding column and at the same time add the last to the first?

231. What change does an nth-order determinant undergo if its matrix is turned through 90° about the "centre"?

232. What is the determinant whose sum of rows with even numbers is equal to the sum of rows with odd numbers?

233. Find the sum of all determinants of order $n \geqslant 2$, in each of which one element in each row and each column is equal to unity, the others being zero. How many such determinants are there in all?

234. Find the sum of the determinants of order $n \geqslant 2$:

$$\sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = 1}^{n} \begin{vmatrix} a_{1\alpha_1} & a_{1\alpha_2} & \ldots & a_{1\alpha_n} \\ a_{2\alpha_1} & a_{2\alpha_2} & \ldots & a_{2\alpha_n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n\alpha_1} & a_{n\alpha_2} & \ldots & a_{n\alpha_n} \end{vmatrix},$$

where the sum is taken over all values $\alpha_1, \alpha_2, \ldots, \alpha_n$ that vary from 1 to n independently.

235. Let all elements of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

be single-valued integers. Denote by N_i the number written by the digits of the ith row of the determinant with the arrangement retained (a_{in} is the number of units, $a_{i, n-1}$ the number of tens, and so on). Prove that the value of the determinant is divisible by the greatest common divisor of the numbers $N_1, N_2, \dots, N_n$.

236. Compute the following determinant by expanding in terms of the third row:

$$\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}.$$

237. Expanding in terms of the second column, compute the determinant

$$\begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix}.$$

Compute the following determinants:

238. $\begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix}$.

239. $\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$.

240. $\begin{vmatrix} x & a & b & 0 & c \\ 0 & y & 0 & 0 & d \\ 0 & e & z & 0 & f \\ g & h & k & u & l \\ 0 & 0 & 0 & 0 & v \end{vmatrix}$.

241. Let M_{ij} be a minor of element a_{ij} of determinant D. Show that if D is a symmetric determinant or a skew-symmetric determinant of odd order, then $M_{ij} = M_{ji}$; but if D is a skew-symmetric determinant of even order, then $M_{ij} = -M_{ji}$.

242. Let D be a determinant of order $n > 1$, D' and D'' determinants obtained from D by replacing each element a_{ij}

by its cofactor A_{ij} for D' and by its minor M_{ij} for D''. Prove that $D' = D''$. The determinant D' is said to be the adjugate determinant of D. For expressing D' in terms of D see problem 506.

243. Compute the following determinant without expanding it:

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{1} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}.$$

Prove the following identities without expanding the determinants:

***244.**
$$\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}.$$

***245.**
$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-2} & a_1^n \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} & a_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} & a_n^n \end{vmatrix}$$
$$= (a_1 + a_2 + \dots + a_n) \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} & a_n^{n-1} \end{vmatrix}.$$

***246.**
$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{i-1} & a_1^{i+1} & \dots & a_1^n \\ 1 & a_2 & a_2^2 & \dots & a_2^{i-1} & a_2^{i+1} & \dots & a_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{i-1} & a_n^{i+1} & \dots & a_n^n \end{vmatrix}$$
$$= \Big(\sum_{k_1, k_2, \dots k_{n-i}} a_{k_1} a_{k_2} \dots a_{k_{n-i}} \Big) \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix},$$

where the sum is taken over all combinations of n numbers 1, 2, 3, . . ., n with respect to $n - i$.

Using the properties of determinants, including expansions in terms of a row or column, prove the following identities:

*247. $$\begin{vmatrix} \cos\frac{\alpha-\beta}{2} & \sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2} \\ \cos\frac{\beta-\gamma}{2} & \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2} \\ \cos\frac{\gamma-\alpha}{2} & \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2} \end{vmatrix} = \frac{1}{2}[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\gamma)].$$

248. $$\begin{vmatrix} \sin^2\alpha & \sin\alpha\cos\alpha & \cos^2\alpha \\ \sin^2\beta & \sin\beta\cos\beta & \cos^2\beta \\ \sin^2\gamma & \sin\gamma\cos\gamma & \cos^2\gamma \end{vmatrix} = \sin(\alpha-\beta)\cos\alpha\cos\beta + \sin(\beta-\gamma)\cos\beta\cos\gamma + \sin(\gamma-\alpha)\cos\gamma\cos\alpha.$$

249. $$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

250. $$\begin{vmatrix} \frac{1}{a+x} & \frac{1}{a+y} & 1 \\ \frac{1}{b+x} & \frac{1}{b+y} & 1 \\ \frac{1}{c+x} & \frac{1}{c+y} & 1 \end{vmatrix} = \frac{(a-b)(a-c)(b-c)(x-y)}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)}.$$

251. $$\begin{vmatrix} \frac{1}{a+x} & \frac{1}{a+y} & \frac{1}{a+z} \\ \frac{1}{b+x} & \frac{1}{b+y} & \frac{1}{b+z} \\ \frac{1}{c+x} & \frac{1}{c+y} & \frac{1}{c+z} \end{vmatrix} = \frac{(a-b)(a-c)(b-c)(x-y)(x-z)(y-z)}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)(a+z)(b+z)(c+z)}.$$

252.

$$\begin{vmatrix} a^2+(1-a^2)\cos\varphi & ab(1-\cos\varphi) & ac(1-\cos\varphi) \\ ba(1-\cos\varphi) & b^2+(1-b^2)\cos\varphi & bc(1-\cos\varphi) \\ ca(1-\cos\varphi) & cb(1-\cos\varphi) & c^2+(1-c^2)\cos\varphi \end{vmatrix} = \cos^2\varphi,$$

when $a^2+b^2+c^2=1$.

***253.**

$$\begin{vmatrix} \cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\theta & -\sin\varphi\cos\psi-\cos\varphi\sin\psi\cos\theta & \sin\psi\sin\theta \\ \cos\varphi\sin\psi+\sin\varphi\cos\psi\cos\theta & -\sin\varphi\sin\psi+\cos\varphi\cos\psi\cos\theta & -\cos\psi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \end{vmatrix} = 1$$

254.

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

255.

$$\begin{vmatrix} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \\ a & b & c & d \end{vmatrix} = a^2+b^2+c^2-2ab-2bc-2ac+2d.$$

***256.**

$$\begin{vmatrix} -1 & 1 & 1 & 1 & x \\ 1 & -1 & 1 & 1 & y \\ 1 & 1 & -1 & 1 & z \\ 1 & 1 & 1 & -1 & u \\ x & y & z & u & 0 \end{vmatrix}$$

$$= -4[x^2+y^2+z^2+u^2-2(xy+xz+xu+yz+yu+zu)].$$

Sec. 4. Evaluating Determinants with Numerical Elements

Compute the following determinants:

257. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$.

258. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$.

259. $\begin{vmatrix} 2 & -5 & 1 & 2 \\ -3 & 7 & -1 & 4 \\ 5 & -9 & 2 & 7 \\ 4 & -6 & 1 & 2 \end{vmatrix}$.

260. $\begin{vmatrix} -3 & 9 & 3 & 6 \\ -5 & 8 & 2 & 7 \\ 4 & -5 & -3 & -2 \\ 7 & -8 & -4 & -5 \end{vmatrix}$.

261. $\begin{vmatrix} 3 & -3 & -5 & 8 \\ -3 & 2 & 4 & -6 \\ 2 & -5 & -7 & 5 \\ -4 & 3 & 5 & -6 \end{vmatrix}$.

262. $\begin{vmatrix} 2 & -5 & 4 & 3 \\ 3 & -4 & 7 & 5 \\ 4 & -9 & 8 & 5 \\ -3 & 2 & -5 & 3 \end{vmatrix}$.

263. $\begin{vmatrix} 3 & -3 & -2 & -5 \\ 2 & 5 & 4 & 6 \\ 5 & 5 & 8 & 7 \\ 4 & 4 & 5 & 6 \end{vmatrix}$.

264. $\begin{vmatrix} 3 & -5 & -2 & 2 \\ -4 & 7 & 4 & 4 \\ 4 & -9 & -3 & 7 \\ 2 & -6 & -3 & 2 \end{vmatrix}$.

265. $\begin{vmatrix} 3 & -5 & 2 & -4 \\ -3 & 4 & -5 & 3 \\ -5 & 7 & -7 & 5 \\ 8 & -8 & 5 & -6 \end{vmatrix}$.

266. $\begin{vmatrix} 3 & 2 & 2 & 2 \\ 9 & -8 & 5 & 10 \\ 5 & -8 & 5 & 8 \\ 6 & -5 & 4 & 7 \end{vmatrix}$.

267. $\begin{vmatrix} 7 & 6 & 3 & 7 \\ 3 & 5 & 7 & 2 \\ 5 & 4 & 3 & 5 \\ 5 & 6 & 5 & 4 \end{vmatrix}$.

268. $\begin{vmatrix} 6 & -5 & 8 & 4 \\ 9 & 7 & 5 & 2 \\ 7 & 5 & 3 & 7 \\ -4 & 8 & -8 & -3 \end{vmatrix}$.

269. $\begin{vmatrix} 7 & 3 & 2 & 6 \\ 8 & -9 & 4 & 9 \\ 7 & -2 & 7 & 3 \\ 5 & -3 & 3 & 4 \end{vmatrix}$.

270. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 7 & 10 & 13 \\ 3 & 5 & 11 & 16 & 21 \\ 2 & -7 & 7 & 7 & 2 \\ 1 & 4 & 5 & 3 & 10 \end{vmatrix}$.

271. $\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 5 & 9 & 7 & 8 & 6 \\ 6 & 12 & 13 & 9 & 7 \\ 4 & 6 & 6 & 5 & 4 \\ 2 & 5 & 4 & 5 & 3 \end{vmatrix}$.

272. $\begin{vmatrix} 35 & 59 & 71 & 52 \\ 42 & 70 & 77 & 54 \\ 43 & 68 & 72 & 52 \\ 29 & 49 & 65 & 50 \end{vmatrix}$.

273. $\begin{vmatrix} 27 & 44 & 40 & 55 \\ 20 & 64 & 21 & 40 \\ 13 & -20 & -13 & 24 \\ 46 & 45 & -55 & 84 \end{vmatrix}$.

274. $\begin{vmatrix} 24 & 11 & 13 & 17 & 19 \\ 51 & 13 & 32 & 40 & 46 \\ 61 & 11 & 14 & 50 & 56 \\ 62 & 20 & 7 & 13 & 52 \\ 80 & 24 & 45 & 57 & 70 \end{vmatrix}$.

275. $\begin{vmatrix} \frac{3}{2} & -\frac{9}{2} & -\frac{3}{2} & -3 \\ \frac{5}{3} & -\frac{8}{3} & -\frac{2}{3} & -\frac{7}{3} \\ \frac{4}{3} & -\frac{5}{3} & -1 & -\frac{2}{3} \\ 7 & -8 & -4 & -5 \end{vmatrix}$.

***276.** $\begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ \frac{2}{3} & -\frac{9}{2} & \frac{4}{5} & \frac{5}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix}$.

277. $\begin{vmatrix} \frac{3}{4} & 2 & -\frac{1}{2} & -5 \\ 1 & -2 & \frac{3}{2} & 8 \\ \frac{5}{6} & -\frac{4}{3} & \frac{4}{3} & \frac{14}{3} \\ \frac{2}{5} & -\frac{4}{5} & \frac{1}{2} & \frac{12}{5} \end{vmatrix}$.

278. $$\begin{vmatrix} \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{3} \\ \sqrt{6} & \sqrt{21} & \sqrt{10} & -2\sqrt{3} \\ \sqrt{10} & 2\sqrt{15} & 5 & \sqrt{6} \\ 2 & 2\sqrt{6} & \sqrt{10} & \sqrt{15} \end{vmatrix}.$$

Sec. 5. Methods of Computing Determinants of the nth Order

Introduction. The approach to computing determinants with numerical elements, which consists in making all elements of a certain row (or column) except one vanish with a subsequent reduction in the order, becomes extremely unwieldy in the case of determinants of the given order with literal elements. In the general case, this leads to an expression like that obtained when computing a determinant by direct application of the definition of a determinant. This method is all the more inconvenient when dealing with determinants of arbitrary order with literal or numerical elements.

There is no general method for computing such determinants (with the exception of the expression given in the definition of a determinant). In handling special types of determinants, one makes use of a variety of methods of computation that lead to simpler expressions of the determinant (that is, such that contain a smaller number of operations) than that obtained from the definition. We will now investigate several of the most widely used methods and will then offer problems in each method and also problems in which the student himself chooses the most suitable method of solution. To simplify orientation, problems involving the Laplace theorem and multiplication of determinants are collected in separate sections.

1. Method of reducing to triangular form. This method consists in transforming the determinant to a form in which all elements to one side of a diagonal are equal to zero. The case of the secondary diagonal can be reduced to that of the principal diagonal by reversing the order of the rows (or columns). The resulting determinant is equal to the product of the elements of the principal diagonal.

Example 1. Compute the following determinant of order n:

$$D=\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}.$$

Subtract the first row from all the others:

$$D=\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}=(-1)^{n-1}.$$

Example 2. Compute the determinant

$$D=\begin{vmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & a_n \end{vmatrix}.$$

Subtract the first row from all the others:

$$D=\begin{vmatrix} a_1 & x & x & \dots & x \\ x-a_1 & a_2-x & 0 & \dots & 0 \\ x-a_1 & 0 & a_3-x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ x-a_1 & 0 & 0 & \dots & a_n-x \end{vmatrix}.$$

Take out a_1-x from the first column, $a_2-x\dots$ from the second and so on, and a_n-x from the nth:

$$D=(a_1-x)(a_2-x)\dots(a_n-x)$$

$$\times\begin{vmatrix} \frac{a_1}{a_1-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \dots & \frac{x}{a_n-x} \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & & 1 \end{vmatrix}.$$

Put $\frac{a_1}{a_1-x}=1+\frac{x}{a_1-x}$ and add all columns to the first one:

$$D=(a_1-x)(a_2-x)\ldots(a_n-x)$$

$$\times\begin{vmatrix} 1+\frac{x}{a_1-x}+\ldots+\frac{x}{a_n-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \ldots & \frac{x}{a_n-x} \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{vmatrix}$$

$$=x(a_1-x)(a_2-x)\ldots(a_n-x)$$

$$\times\left(\frac{1}{x}+\frac{1}{a_1-x}+\frac{1}{a_2-x}+\ldots+\frac{1}{a_n-x}\right).$$

2. Method of isolating linear factors. The determinant is regarded as a polynomial in one or several component letters. Transforming it, one finds that it is divisible by a series of linear factors and, hence (if these factors are coprime), by their product.

By comparing the separate terms of the determinant with the terms of the product of linear factors, one finds the quotient obtained from the division of the determinant by this product and, hence, the expression of the determinant.

Example 3. Compute the determinant

$$D=\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$$

Adding all columns to the first column, we find that the determinant is divisible by $x+y+z$; adding to the first column the second column and subtracting the third and fourth yields the factor $y+z-x$; adding the third column to the first and subtracting the second and fourth columns yields the factor $x-y+z$; finally, adding the fourth column to the first and subtracting the second and third yields the factor $x+y-z$. Regarding x, y, z as independent unknowns, we conclude that all these four factors are relatively prime in pairs and, hence, the determinant is divisible by their product $(x+y+z)(y+z-x)\times$ $\times(x-y+z)(x+y-z)$.

This product contains the term z^4 with the coefficient -1, while the determinant itself contains the same term z^4 with the coefficient $+1$. Thus

$$D = -(x + y + z)(y + z - x)(x + z - y)(x + y - z)$$
$$= x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2.$$

Example 4. Using the method of isolating linear factors, compute the following Vandermonde determinant of the nth order:

$$D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}.$$

Regarding D_n as a polynomial in the single unknown x_n with coefficients depending on $x_1, \dots, x_{n-1}$, we see that it vanishes for $x_n = x_1$, $x_n = x_2$, $\dots$, $x_n = x_{n-1}$ and therefore is divisible by $x_n - x_1$, $x_n - x_2$, $\dots$, $x_n - x_{n-1}$.

All these factors are coprime (since $x_1, x_2, \dots, x_n$ are algebraically independent).

And so D_n is divisible by their product, that is,

$$D_n = q(x_1, x_2, \dots, x_n)(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1}).$$

Expanding D_n in terms of the last row, we see that it is a polynomial of degree $n - 1$ in x_n, and the coefficient of x_n^{n-1} is equal to the Vandermonde determinant D_{n-1} of the unknowns $x_1, x_2, \dots, x_{n-1}$; since the product of the parentheses in the right member of the last equation contains x_n^{n-1} with the coefficient 1, the polynomial $q(x_1, x_2, \dots, x_n)$ does not contain x_n and, comparing the coefficients of x_n^{n-1} in both members of the equation, we get $D_{n-1} = q(x_1, x_2, \dots, x_{n-1})$, whence $D_n = D_{n-1}(x_n - x_1) \times (x_n - x_2) \dots (x_n - x_{n-1})$. Applying this equation with $n - 1$ substituted for n, we have

$$D_{n-1} = D_{n-2}(x_{n-1} - x_1) \dots (x_{n-1} - x_{n-2}).$$

Now put this expression for D_{n-1} into the preceding expression for D_n. Repeating the reasoning, we will finally isolate the factor $x_2 - x_1$ and then arrive at a Vandermonde determinant of the first order $D_1 = 1$.

Thus

$$D_n = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \ldots (x_n - x_1)(x_n - x_2) \ldots (x_n - x_{n-1}) = \prod_{i>j} (x_i - x_j).$$

3. The method of recurrence relations. This method consists in transforming the given determinant and expanding it in terms of a row or a column and thus expressing it in terms of determinants of the same type but of lower order. The result is called a recurrence relation.

Then, using the general form of the determinant, one computes directly as many determinants of lower order as there were in the right-hand member of the recurrence relation. The higher-order determinants are computed in succession from the recurrence relation. If one has to obtain an expression for a determinant of any order n, then, by computing from the recurrence relation several lower-order determinants, one attempts to notice the general form of the desired expression and then proves the validity of that expression for arbitrary n with the aid of the recurrence relation and by induction on n.

The general expression can be obtained in another way. Substitute into the recurrence relation that expresses an nth-order determinant the expression of the determinant of order $n - 1$ from the very same recurrence relation with $n - 1$ substituted for n; a similar expression of the determinant of order $n - 2$ is substituted and so on until the form of the desired general expression for the nth-order determinant is obtained. It is also possible to combine both approaches, using the latter to detect the desired expression and then proving the validity of that expression via induction on n. The method of recurrence relations is the strongest of the methods investigated here and is applicable to the more complicated determinants.

Before examining some examples in computing determinants by the method of recurrence relations, let us investigate one more special case where the recurrence relation yields an algorithm for problem solving that eliminates the guesswork inherent in the general case. Suppose we have a recurrence relation of the form

$$D_n = pD_{n-1} + qD_{n-2}, \quad n > 2 \tag{1}$$

where p, q are constants, that is, quantities independent of n.*

For $q = 0$, D_n is computed as a term of a geometric progression: $D_n = p^{n-1}D_1$. Here D_1 is a first-order determinant of the given type, that is to say, an element of the determinant D_n lying in the upper left-hand corner.

Let $q \neq 0$ and let α, β be the roots of the quadratic equation $x^2 - px - q = 0$. Then $p = \alpha + \beta$, $q = -\alpha\beta$ and equation (1) may be rewritten thus:

$$D_n - \beta D_{n-1} = \alpha (D_{n-1} - \beta D_{n-2}) \tag{2}$$

or

$$D_n - \alpha D_{n-1} = \beta (D_{n-1} - \alpha D_{n-2}). \tag{3}$$

First assume that $\alpha \neq \beta$.

Using the formula for the $(n-1)$th term of the geometric progression, we find, from (2) and (3),

$$D_n - \beta D_{n-1} = \alpha^{n-2}(D_2 - \beta D_1) \text{ and}$$

$$D_n - \alpha D_{n-1} = \beta^{n-2}(D_2 - \alpha D_1)$$

whence

$$D_n = \frac{\alpha^{n-1}(D_2 - \beta D_1) - \beta^{n-1}(D_2 - \alpha D_1)}{\alpha - \beta}$$

or

$$D_n = C_1\alpha^n + C_2\beta^n, \text{ where } C_1 = \frac{D_2 - \beta D_1}{\alpha(\alpha - \beta)}, \quad C_2 = -\frac{D_2 - \alpha D_1}{\beta(\alpha - \beta)}. \tag{4}$$

The last expression for D_n can easily be remembered. It was derived for $n > 2$, but can be verified directly for $n = 1$ and $n = 2$. The values of C_1 and C_2 can be found from the initial conditions $D_1 = C_1\alpha + C_2\beta$, $D_2 = C_1\alpha^2 + C_2\beta^2$ instead of from the expressions (4).

Now let $\alpha = \beta$. Then (2) and (3) transform to

$$D_n - \alpha D_{n-1} = \alpha (D_{n-1} - \alpha D_{n-2}),$$

whence

$$D_n - \alpha D_{n-1} = A\alpha^{n-2}, \tag{5}$$

where

$$A = D_2 - \alpha D_1.$$

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* This method was suggested to the author by L.Ya. Okunev. It is also applicable to the recurrence relation $D_n = p_1 D_{n-1} + \ldots + p_k D_{n-k}$ with constant $p_1, \ldots, p_k$ and arbitrary k, but we confine ourselves to $k = 2$ since the reasoning is extremely involved.

Replacing n by $n-1$, we get $D_{n-1} - \alpha D_{n-2} = A\alpha^{n-3}$, whence $D_{n-1} = \alpha D_{n-2} + A\alpha^{n-3}$. Putting this expression into (5), we get $D_n = \alpha^2 D_{n-2} + 2A\alpha^{n-2}$. Repeating this same device several times, we obtain $D_n = \alpha^{n-1} D_1 + (n-1) A\alpha^{n-2}$ or $D_n = \alpha^n [(n-1) C_1 + C_2]$, where $C_1 = \frac{A}{\alpha^2}$, $C_2 = \frac{D_1}{\alpha}$ (here $\alpha \neq 0$, since $q \neq 0$).

Example 5. Use the method of recurrence relations to compute the determinant in example 2.

Expressing the element in the lower right-hand corner as $a_n = x + (a_n - x)$, we can decompose the determinant D_n into a sum of two determinants:

$$D_n = \begin{vmatrix} a_1 & x & \dots & x & x \\ x & a_2 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & a_{n-1} & x \\ x & x & \dots & x & x \end{vmatrix} + \begin{vmatrix} a_1 & x & \dots & x & 0 \\ x & a_2 & \dots & x & 0 \\ \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & a_{n-1} & 0 \\ x & x & \dots & x & a_n - x \end{vmatrix}.$$

In the first determinant, subtract the last column from all the others, and then expand the second determinant in terms of the last column:

$$D_n = x(a_1 - x)(a_2 - x) \dots (a_{n-1} - x) + (a_n - x) D_{n-1}$$

This then is a recurrence relation. Putting into it an analogous expression for D_{n-1}, we get

$$\begin{aligned} D_n = {} & x(a_1 - x)(a_2 - x) \dots (a_{n-1} - x) \\ & + x(a_1 - x)(a_2 - x) \dots (a_{n-2} - x)(a_n - x) \\ & + D_{n-2}(a_{n-1} - x)(a_n - x). \end{aligned}$$

Repeating this reasoning $n-1$ times and noting that $D_1 = a_1 = x + (a_1 - x)$, we obtain

$$\begin{aligned} D_n = {} & x(a_1 - x)(a_2 - x) \dots (a_{n-1} - x) \\ & + x(a_1 - x) \dots (a_{n-2} - x)(a_n - x) \\ & \dots + x(a_2 - x) \dots (a_n - x) + (a_1 - x)(a_2 - x) \dots (a_n - x) \\ = {} & x(a_1 - x)(a_2 - x) \dots (a_n - x) \left(\frac{1}{x} + \frac{1}{a_1 - x} + \dots + \frac{1}{a_n - x} \right), \end{aligned}$$

which coincides with the result of example 2.

Example 6. Compute the nth-order determinant

$$D_n = \begin{vmatrix} 5 & 3 & 0 & 0 & \dots & 0 & 0 \\ 2 & 5 & 3 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & 5 \end{vmatrix}.$$

Expanding by the first row, we find the recurrence relation

$$D_n = 5D_{n-1} - 6D_{n-2}.$$

The equation $x^2 - 5x + 6 = 0$ has the roots $\alpha = 2$, $\beta = 3$.

By formula (4), $D_n = C_1\alpha^n + C_2\beta^n = 3^{n+1} - 2^{n+1}$.

4. The method of representing a determinant as a sum of determinants. Some determinants are readily evaluated by decomposing them into a sum of determinants of the same order with respect to rows (or columns).

Example 7. Compute the determinant

$$D_n = \begin{vmatrix} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \dots & \dots & \dots & \dots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix}.$$

With respect to the first row, this determinant can be decomposed into two determinants, each of which can be decomposed into two determinants with respect to the second row, and so on. When we reach the last row, we have 2^n determinants.

If in each decomposition we take for the first addends the numbers a_i and for the second addends the numbers b_j, then the rows of the resulting determinants will be either of the form $a_i, a_i, \dots, a_i$ or of the form $b_1, b_2, \dots, b_n$. Two rows of the first type are proportional, while those of the second type are equal. When $n > 2$, at least two rows of one type appear in each resulting determinant, and it vanishes. Thus, $D_n = 0$ for $n > 2$.

Then

$$D_1 = a_1 + b_1, \quad D_2 = \begin{vmatrix} a_1 & a_1 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ a_2 & a_2 \end{vmatrix} = (a_1 - a_2)(b_2 - b_1).$$

5. The method of changing the elements of a determinant. This method is applied when changing all elements of a determinant by the same number yields a form such that the cofactors of all elements can readily be computed. The method is based on the following property: if the same number x is added to all elements of the determinant D, the determinant is increased by the product of the number x into the sum of the cofactors of all elements of D. True enough, let

$$D=\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}, \quad D'=\begin{vmatrix} a_{11}+x & \dots & a_{1n}+x \\ \dots & \dots & \dots \\ a_{n1}+x & \dots & a_{nn}+x \end{vmatrix}.$$

Decompose D' into two determinants with respect to the first row, then each of them into two determinants with respect to the second row, and so on.

The terms containing more than one row of elements equal to x are equal to zero.

The terms containing one row of elements equal to x are then expanded in terms of that row to get $D' = D + x \sum_{i, j=1}^{n} A_{ij}$, which is what we sought.

Thus, computing the determinant D' reduces to computing D and the sum of its cofactors.

Example 8. Compute the determinant D_n of example 2.

Subtracting x from all its elements, we obtain the determinant

$$D=\begin{vmatrix} a_1-x & 0 & \dots & 0 \\ 0 & a_2-x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n-x \end{vmatrix}.$$

The cofactors of the elements of D off the principal diagonal are equal to zero, and the cofactor of each element on the principal diagonal is equal to the product of the remaining elements of the principal diagonal. Therefore

$$D_n=(a_1-x)\dots(a_n-x)$$
$$+x\sum_{i=1}^{n}(a_1-x)\dots(a_{i-1}-x)(a_{i+1}-x)\dots(a_n-x)$$
$$=x(a_1-x)(a_2-x)\dots(a_n-x)\left(\frac{1}{x}+\frac{1}{a_1-x}+\dots+\frac{1}{a_n-x}\right).$$

Evaluate the following determinants by reduction to triangular form (wherever the order of the determinant cannot be judged exactly, it is assumed to be of order n):

279.
$$\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}.$$

280.
$$\begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & n \\ 3 & 4 & 5 & \dots & n & n & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n & n & n & \dots & n & n & n \end{vmatrix}.$$

281.
$$\begin{vmatrix} x_1 & a_{12} & a_{13} & \dots & a_{1n} \\ x_1 & x_2 & a_{23} & \dots & a_{2n} \\ x_1 & x_2 & x_3 & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ x_1 & x_2 & x_3 & \dots & x_n \end{vmatrix}.$$

282.
$$\begin{vmatrix} 1 & \dots & 1 & 1 & 1 \\ a_1 & \dots & a_1 & a_1-b_1 & a_1 \\ a_2 & \dots & a_2-b_2 & a_2 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_n-b_n & \dots & a_n & a_n & a_n \end{vmatrix}.$$

283.
$$\begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ 2 & 3 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & 3 \end{vmatrix}.$$

284.
$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -x & x & 0 & \dots & 0 \\ 0 & -x & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix}.$$

***285.**
$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -x_1 & x_2 & 0 & \dots & 0 \\ 0 & -x_2 & x_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x_n \end{vmatrix}.$$

286. Compute a determinant of order n whose elements are given by the conditions $a_{ij} = \min(i, j)$.

287. Compute a determinant of order n whose elements are given by $a_{ij} = \max(i, j)$.

***288.** Compute a determinant of order n whose elements are given by $a_{ij} = |i - j|$.

Evaluate the following determinants using the method of isolation of linear factors:

289.
$$\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 1 & x+1 & 3 & \dots & n \\ 1 & 2 & x+1 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & x+1 \end{vmatrix}.$$

290.
$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2-x & 1 & \dots & 1 \\ 1 & 1 & 3-x & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & n+1-x \end{vmatrix}$$

291.
$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_0 & x & a_2 & \dots & a_n \\ a_0 & a_1 & x & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & a_2 & \dots & x \end{vmatrix}.$$

292.
$$\begin{vmatrix} -x & a & b & c \\ a & -x & c & b \\ b & c & -x & a \\ \dots & \dots & \dots & \dots \\ c & b & a & -x \end{vmatrix}.$$

293.
$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}.$$

***294.**
$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix}.$$

Compute the following determinants by the method of recurrence relations:

*295. $\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \dots & a_1b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \dots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \dots & a_3b_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1b_n & a_2b_n & a_3b_n & \dots & a_nb_n \end{vmatrix}$.

*296. $\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -y_1 & x_1 & 0 & \dots & 0 \\ 0 & -y_2 & x_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x_n \end{vmatrix}$.

297. $\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & \dots & 0 \\ 1 & 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & a_n \end{vmatrix}$.

298. $\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & a_2 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & 1 & a_n \end{vmatrix}$.

299. $\begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 \end{vmatrix}$.

300. $\begin{vmatrix} 3 & 2 & 0 & \dots & 0 \\ 1 & 3 & 2 & \dots & 0 \\ 0 & 1 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 3 \end{vmatrix}$.

301. $\begin{vmatrix} 7 & 5 & 0 & \dots & 0 \\ 2 & 7 & 5 & \dots & 0 \\ 0 & 2 & 7 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 7 \end{vmatrix}$.

302. $\begin{vmatrix} 5 & 6 & 0 & 0 & 0 & \dots & 0 & 0 \\ 4 & 5 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix}$.

303.
$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3 & 4 & 3 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 5 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 5 \end{vmatrix}.$$

304.
$$\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & 0 & \dots & 0 \\ 1 & \alpha+\beta & \alpha\beta & 0 & \dots & 0 \\ 0 & 1 & \alpha+\beta & \alpha\beta & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha+\beta \end{vmatrix}.$$

Compute the following determinants by representing them as a sum of determinants:

***305.**
$$\begin{vmatrix} x+a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & x+a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & x+a_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x+a_n \end{vmatrix}.$$

***306.**
$$\begin{vmatrix} x_1 & a_2 & \dots & a_n \\ a_1 & x_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & x_n \end{vmatrix}.$$

***307.**
$$\begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & 0 & \dots & 0 & 0 \\ x_3 & x_3 & x_3 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_n & x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix}.$$

308.
$$\begin{vmatrix} x_1 & a_1b_2 & a_1b_3 & \dots & a_1b_n \\ a_2b_1 & x_2 & a_2b_3 & \dots & a_2b_n \\ a_3b_1 & a_3b_2 & x_3 & \dots & a_3b_n \\ \dots & \dots & \dots & \dots & \dots \\ a_nb_1 & a_nb_2 & a_nb_3 & \dots & x_n \end{vmatrix}.$$

Compute the following determinants (the order of any determinant that is not clear is assumed to be n):

309. $\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 3 & 3 & \dots & n-1 & n \\ 1 & 2 & 5 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & 2n-3 & n \\ 1 & 2 & 3 & \dots & n-1 & 2n-1 \end{vmatrix}$.

310. $\begin{vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1+b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2+b_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & a_n+b_n \end{vmatrix}$.

311. $\begin{vmatrix} 2 & 2 & \dots & 2 & 2 & 1 \\ 2 & 2 & \dots & 2 & 2 & 2 \\ 2 & 2 & \dots & 3 & 2 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & n-1 & \dots & 2 & 2 & 2 \\ n & 2 & \dots & 2 & 2 & 2 \end{vmatrix}$.

312. $\begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ n & n & n & \dots & n \end{vmatrix}$.

313. $\begin{vmatrix} x & y & 0 & 0 & \dots & 0 & 0 \\ 0 & x & y & 0 & \dots & 0 & 0 \\ 0 & 0 & x & y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}$.

314. $\begin{vmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ 1 & 1 & 1-n & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1-n \end{vmatrix}$.

315. $\begin{vmatrix} 1 & 1 & \dots & 1 & -n \\ 1 & 1 & \dots & -n & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & -n & \dots & 1 & 1 \\ -n & 1 & \dots & 1 & 1 \end{vmatrix}$.

316. $\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}$.

317. $\begin{vmatrix} n & 1 & 1 & \dots & 1 \\ 1 & n & 1 & \dots & 1 \\ 1 & 1 & n & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & n \end{vmatrix}$.

318. $\begin{vmatrix} a & b & \dots & b & b \\ b & a & \dots & b & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & \dots & a & b \\ b & b & \dots & b & a \end{vmatrix}$.

319. $\begin{vmatrix} 1 & 2 & 0 & 0 & \dots & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 3 \end{vmatrix}$.

320. $\begin{vmatrix} 1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1-b_1 & b_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_2 & b_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{vmatrix}$.

321. $\begin{vmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & 1 \end{vmatrix}$.

322. $\begin{vmatrix} a_0 & -1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & x & -1 & 0 & \dots & 0 & 0 \\ a_2 & 0 & x & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & 0 & \dots & x & -1 \\ a_n & 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}$.

323.
$$\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ -1 & x & 0 & \dots & 0 & 0 \\ 0 & -1 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & \dots & -1 & x \end{vmatrix}.$$

324.
$$\begin{vmatrix} n & -1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & x & -1 & 0 & \dots & 0 & 0 \\ n-2 & 0 & x & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 0 & 0 & 0 & \dots & x & -1 \\ 1 & 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}.$$

325.
$$\begin{vmatrix} 1 & x & x^2 & x^3 & \dots & x^n \\ a_{11} & 1 & x & x^2 & \dots & x^{n-1} \\ a_{21} & a_{22} & 1 & x & \dots & x^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots & 1 \end{vmatrix}.$$

326.
$$\begin{vmatrix} 1 & C_n^1 & C_n^2 & \dots & C_n^{n-1} & C_n^n \\ 1 & C_{n-1}^1 & C_{n-1}^2 & \dots & C_{n-1}^{n-1} & 0 \\ 1 & C_{n-2}^1 & C_{n-2}^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & C_2^1 & C_2^2 & \dots & 0 & 0 \\ 1 & C_1^1 & 0 & \dots & 0 & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \end{vmatrix}.$$

327.
$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & x & \dots & x & x \\ 1 & x & 0 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & \dots & 0 & x \\ 1 & x & x & \dots & x & 0 \end{vmatrix}.$$

328. $$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^n \\ 1 & 3 & 3^2 & \dots & 3^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n+1 & (n+1)^2 & \dots & (n+1)^n \end{vmatrix}.$$

329. $$\begin{vmatrix} a^n & (a-1)^n & \dots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \dots & (a-n)^{n-1} \\ \dots & \dots & \dots & \dots \\ a & a-1 & \dots & a-n \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

330. $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1+1 & x_2+1 & \dots & x_n+1 \\ x_1^2+x_1 & x_2^2+x_2 & \dots & x_n^2+x_n \\ x_1^3+x_1^2 & x_2^3+x_2^2 & \dots & x_n^3+x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1}+x_1^{n-2} & x_2^{n-1}+x_2^{n-2} & \dots & x_n^{n-1}+x_n^{n-2} \end{vmatrix}.$$

331. $$\begin{vmatrix} (x+a_1)^n & (x+a_1)^{n-1} & \dots & x+a_1 & 1 \\ (x+a_2)^n & (x+a_2)^{n-1} & \dots & x+a_2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ (x+a_{n+1})^n & (x+a_{n+1})^{n-1} & \dots & x+a_{n+1} & 1 \end{vmatrix}.$$

332. $$\begin{vmatrix} 1 & \sin\varphi_1 & \sin^2\varphi_1 & \dots & \sin^{n-1}\varphi_1 \\ 1 & \sin\varphi_2 & \sin^2\varphi_2 & \dots & \sin^{n-1}\varphi_2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \sin\varphi_n & \sin^2\varphi_n & \dots & \sin^{n-1}\varphi_n \end{vmatrix}.$$

333. $$\begin{vmatrix} 1 & \cos\varphi_1 & \cos^2\varphi_1 & \dots & \cos^{n-1}\varphi_1 \\ 1 & \cos\varphi_2 & \cos^2\varphi_2 & \dots & \cos^{n-1}\varphi_2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\varphi_n & \cos^2\varphi_n & \dots & \cos^{n-1}\varphi_n \end{vmatrix}.$$

334. $$\begin{vmatrix} 1 & \varphi_1(x_1) & \varphi_2(x_1) & \dots & \varphi_{n-1}(x_1) \\ 1 & \varphi_1(x_2) & \varphi_2(x_2) & \dots & \varphi_{n-1}(x_2) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varphi_1(x_n) & \varphi_2(x_n) & \dots & \varphi_{n-1}(x_n) \end{vmatrix},$$

where $\varphi_k(x) = x^k + a_{k1}x^{k-1} + a_{k2}x^{k-2} + \ldots + a_{kk}$.

335. $$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ f_1(\cos\varphi_1) & f_1(\cos\varphi_2) & \ldots & f_1(\cos\varphi_n) \\ f_2(\cos\varphi_1) & f_2(\cos\varphi_2) & \ldots & f_2(\cos\varphi_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{n-1}(\cos\varphi_1) & f_{n-1}(\cos\varphi_2) & \ldots & f_{n-1}(\cos\varphi_n) \end{vmatrix},$$

where $f_k(x) = a_{k0}x^k + a_{k1}x^{k-1} + a_{k2}x^{k-2} + \ldots + a_{kk}$.

336. $$\begin{vmatrix} 1 & C_{x_1}^1 & C_{x_1}^2 & \ldots & C_{x_1}^{n-1} \\ 1 & C_{x_2}^1 & C_{x_2}^2 & \ldots & C_{x_2}^{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1 & C_{x_n}^1 & C_{x_n}^2 & \ldots & C_{x_n}^{n-1} \end{vmatrix},$$

where $C_x^k = \dfrac{x(x-1)(x-2)\ldots(x-k+1)}{k!}$.

337. $$\begin{vmatrix} (2n-1)^n & (2n-2)^n & \ldots & n^n & (2n)^n \\ (2n-1)^{n-1} & (2n-2)^{n-1} & \ldots & n^{n-1} & (2n)^{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2n-1 & 2n-2 & \ldots & n & 2n \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix}.$$

338. $$\begin{vmatrix} \dfrac{x_1}{x_1-1} & \dfrac{x_2}{x_2-1} & \ldots & \dfrac{x_n}{x_n-1} \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}.$$

339. $$\begin{vmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2^3 & 3^3 & \ldots & n^3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1 & 2^{2n-1} & 3^{2n-1} & \ldots & n^{2n-1} \end{vmatrix}.$$

340. $$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \ldots & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \ldots & b_2^n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \ldots & b_{n+1}^n \end{vmatrix}.$$

341. $$\begin{vmatrix} \sin^{n-1}\alpha_1 & \sin^{n-2}\alpha_1 & \cos\alpha_1 & \dots & \cos^{n-1}\alpha_1 \\ \sin^{n-1}\alpha_2 & \sin^{n-2}\alpha_2 & \cos\alpha_2 & \dots & \cos^{n-1}\alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ \sin^{n-1}\alpha_n & \sin^{n-2}\alpha_n & \cos\alpha_n & \dots & \cos^{n-1}\alpha_n \end{vmatrix}.$$

342.

$$\begin{vmatrix} f_n(x_1, y_1) & y_1 f_{n-1}(x_1, y_1) & \dots & y_1^{n-1} f_1(x_1, y_1) & y_1^n \\ f_n(x_2, y_2) & y_2 f_{n-1}(x_2, y_2) & \dots & y_2^{n-1} f_1(x_2, y_2) & y_2^n \\ \dots & \dots & \dots & \dots & \dots \\ f_n(x_{n+1}, y_{n+1}) & y_{n+1} f_{n-1}(x_{n+1}, y_{n+1}) & \dots & y_{n+1}^{n-1} f_1(x_{n+1}, y_{n+1}) & y_{n+1}^n \end{vmatrix},$$

where $f_i(x, y)$ is a homogeneous polynomial in x, y of degree i.

*343. $$\begin{vmatrix} a_1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ a_2 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}.$$

*344. $$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^n \end{vmatrix}.$$

345. $$\begin{vmatrix} 1 & x_1^2 & x_1^3 & \dots & x_1^n \\ 1 & x_2^2 & x_2^3 & \dots & x_2^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n^2 & x_n^3 & \dots & x_n^n \end{vmatrix}.$$

346. $$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{s-1} & x_1^{s+1} & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{s-1} & x_2^{s+1} & \dots & x_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{s-1} & x_n^{s+1} & \dots & x_n^n \end{vmatrix}.$$

*347. $$\begin{vmatrix} 1 & x_1(x_1-1) & x_1^2(x_1-1) & \dots & x_1^{n-1}(x_1-1) \\ 1 & x_2(x_2-1) & x_2^2(x_2-1) & \dots & x_2^{n-1}(x_2-1) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n(x_n-1) & x_n^2(x_n-1) & \dots & x_n^{n-1}(x_n-1) \end{vmatrix}.$$

*348. $$\begin{vmatrix} 1+x_1 & 1+x_1^2 & \dots & 1+x_1^n \\ 1+x_2 & 1+x_2^2 & \dots & 1+x_2^n \\ \dots & \dots & \dots & \dots \\ 1+x_n & 1+x_n^2 & \dots & 1+x_n^n \end{vmatrix}.$$

*349. $$\begin{vmatrix} 1 & \cos\varphi_1 & \cos 2\varphi_1 & \dots & \cos(n-1)\varphi_1 \\ 1 & \cos\varphi_2 & \cos 2\varphi_2 & \dots & \cos(n-1)\varphi_2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\varphi_n & \cos 2\varphi_n & \dots & \cos(n-1)\varphi_n \end{vmatrix}.$$

*350. $$\begin{vmatrix} \sin\varphi_1 & \sin 2\varphi_1 & \dots & \sin n\varphi_1 \\ \sin\varphi_2 & \sin 2\varphi_2 & \dots & \sin n\varphi_2 \\ \dots & \dots & \dots & \dots \\ \sin\varphi_n & \sin 2\varphi_n & \dots & \sin n\varphi_n \end{vmatrix}.$$

*351. $$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

352. $$\begin{vmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{vmatrix}.$$

*353. $$\begin{vmatrix} x & a_1 & a_2 & \dots & a_n \\ a_1 & x & a_2 & \dots & a_n \\ a_1 & a_2 & x & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x \end{vmatrix}.$$

354. $$\begin{vmatrix} a_1-b_1 & a_1-b_2 & \dots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \dots & a_2-b_n \\ \dots & \dots & \dots & \dots \\ a_n-b_1 & a_n-b_2 & \dots & a_n-b_n \end{vmatrix}.$$

355. $$\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \dots & \dots & \dots & \dots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}.$$

356. $\begin{vmatrix} f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ f_1(a_1) & f_2(a_2) & \dots & f_2(a_n) \\ \dots & \dots & \dots & \dots \\ f_n(a_1) & f_n(a_2) & \dots & f_n(a_n) \end{vmatrix}$, where $f_i(x)$ is a polynomial of degree not exceeding $n-2$.

357. $\begin{vmatrix} 1+a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2+b_1 & 1+a_2+b_2 & \dots & a_2+b_n \\ \dots & \dots & \dots & \dots \\ a_n+b_1 & a_n+b_2 & \dots & 1+a_n+b_n \end{vmatrix}$.

***358.** $\begin{vmatrix} x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & x_2y_2 & \dots & 1+x_2y_n \\ \dots & \dots & \dots & \dots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & x_ny_n \end{vmatrix}$.

***359.** $\begin{vmatrix} a_1-b_1+x & a_1-b_2 & \dots & a_1-b_n \\ a_2-b_1 & a_2-b_2+x & \dots & a_2-b_n \\ \dots & \dots & \dots & \dots \\ a_n-b_1 & a_n-b_2 & \dots & a_n-b_n+x \end{vmatrix}$.

360. $\begin{vmatrix} x_1+a_1b_1 & a_1b_2 & \dots & a_1b_n \\ a_2b_1 & x_2+a_2b_2 & \dots & a_2b_n \\ \dots & \dots & \dots & \dots \\ a_nb_1 & a_nb_2 & \dots & x_n+a_nb_n \end{vmatrix}$.

***361.** $\begin{vmatrix} a & 0 & \dots & 0 & b \\ 0 & a & \dots & b & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & b & \dots & a & 0 \\ b & 0 & \dots & 0 & a \end{vmatrix}$ (the order of the determinant is equal to $2n$).

362. $\begin{vmatrix} a_1 & 0 & \dots & 0 & b_1 \\ 0 & a_2 & \dots & b_2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & b_{2n-1} & \dots & a_{2n-1} & 0 \\ b_{2n} & 0 & \dots & 0 & a_{2n} \end{vmatrix}$.

363. $\begin{vmatrix} \frac{2}{x} & \frac{1}{x^2} & 0 & 0 & \dots & 0 & 0 \\ 1 & \frac{2}{x} & \frac{1}{x^2} & 0 & \dots & 0 & 0 \\ 0 & 1 & \frac{2}{x} & \frac{1}{x^2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & \frac{2}{x} \end{vmatrix}$.

***364.** $\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{vmatrix}$.

***365.** A Fibonacci sequence (Fibonacci was an Italian mathematician of the 13th century) is a number sequence which begins with the numbers 1, 2 and in which each succeeding number is equal to the sum of the two preceding ones: we have the sequence 1, 2, 3, 5, 8, 13, 21, ...

Prove that the nth term of the Fibonacci sequence is equal to the nth-order determinant:

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \end{vmatrix}.$$

Evaluate the following determinants:

366. $\begin{vmatrix} 9 & 5 & 0 & 0 & \dots & 0 & 0 \\ 4 & 9 & 5 & 0 & \dots & 0 & 0 \\ 0 & 4 & 9 & 5 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 4 & 9 \end{vmatrix}$.

367. $\begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$.

368.
$$\begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix}.$$

***369.**
$$\begin{vmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & a & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & a \end{vmatrix}.$$

370.
$$\begin{vmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & a & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & a \end{vmatrix}.$$

***371.** Prove the following equation:

$$\begin{vmatrix} \cos\alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2\cos\alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2\cos\alpha & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2\cos\alpha \end{vmatrix} = \cos n\alpha.$$

Using this result and the result of problem 369, obtain the expression $\cos n\alpha$ in terms of $\cos\alpha$.

372. Prove the equality

$$\frac{\sin n\alpha}{\sin\alpha} = \begin{vmatrix} 2\cos\alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2\cos\alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2\cos\alpha & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2\cos\alpha \end{vmatrix},$$

where the determinant is of order $n - 1$. Using this equality and the result of problem 369, represent $\sin n\alpha$ as a product of $\sin\alpha$ into a polynomial of $\cos\alpha$.

*373. Prove the following equality without computing the determinants:

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c_{n-1} & a_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1c_1 & 0 & 0 & \dots & 0 & 0 \\ 1 & a_2 & b_2c_2 & 0 & \dots & 0 & 0 \\ 0 & 1 & a_3 & b_3c_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & a_n \end{vmatrix}.$$

Compute the following determinants:

374. $$\begin{vmatrix} 1+x^2 & x & 0 & 0 & \dots & 0 & 0 \\ x & 1+x^2 & x & 0 & \dots & 0 & 0 \\ 0 & x & 1+x^2 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x & 1+x^2 \end{vmatrix}.$$

375. $$\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & n & 1 \\ 3 & 4 & 5 & \dots & 1 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n & 1 & 2 & \dots & n-2 & n-1 \end{vmatrix}.$$

*376.

$$\begin{vmatrix} a & a+x & a+2x & \dots & a+(n-2)x & a+(n-1)x \\ a+(n-1)x & a & a+x & \dots & a+(n-3)x & a+(n-2)x \\ a+(n-2)x & a+(n-1)x & a & \dots & a+(n-4)x & a+(n-3)x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a+x & a+2x & a+3x & \dots & a+(n-1)x & a \end{vmatrix}.$$

377. $$\begin{vmatrix} 1 & x & x^2 & \dots & x^{n-2} & x^{n-1} \\ x^{n-1} & 1 & x & \dots & x^{n-3} & x^{n-2} \\ x^{n-2} & x^{n-1} & 1 & \dots & x^{n-4} & x^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & x^2 & x^3 & \dots & x^{n-1} & 1 \end{vmatrix}.$$

378. Determine, without computing the determinants, how the two circulants are related:

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-3} & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \\ a_3 & a_4 & a_5 & \dots & a_1 & a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{vmatrix}.$$

The circulants are built up of the same numbers $a_1, a_2, \dots, a_n$ by means of circular permutations in two opposite directions.

Compute the following determinants:

***379.**
$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \binom{2}{1} & \binom{3}{1} & \dots & \binom{n}{1} \\ 1 & \binom{3}{2} & \binom{4}{2} & \dots & \binom{n+1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n}{n-1} & \binom{n+1}{n-1} & \dots & \binom{2n-2}{n-1} \end{vmatrix},$$

where $\binom{n}{k} = C_n^k = \frac{n!}{k!\,(n-k)!}$.

***380.**
$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \binom{m}{1} & \binom{m+1}{1} & \binom{m+2}{1} & \dots & \binom{m+n}{1} \\ \binom{m+1}{2} & \binom{m+2}{2} & \binom{m+3}{2} & \dots & \binom{m+n+1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ \binom{m+n-1}{n} & \binom{m+n}{n} & \binom{m+n+1}{n} & \dots & \binom{m+2n-1}{n} \end{vmatrix}.$$

*381. $$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \binom{2}{1} & \binom{2}{2} & 0 & \dots & 0 \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-1} \end{vmatrix}.$$

*382. $$\begin{vmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \dots & \binom{n}{n} \\ 1 & \binom{n+1}{1} & \binom{n+1}{2} & \dots & \binom{n+1}{n} \\ 1 & \binom{n+2}{1} & \binom{n+2}{2} & \dots & \binom{n+2}{n} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{2n}{1} & \binom{2n}{2} & \dots & \binom{2n}{n} \end{vmatrix}.$$

*383. $$\begin{vmatrix} \binom{p+n}{n} & \binom{p+n+1}{n} & \dots & \binom{p+2n}{n} \\ \binom{p+n+1}{n} & \binom{p+n+2}{n} & \dots & \binom{p+2n+1}{n} \\ \dots & \dots & \dots & \dots \\ \binom{p+2n}{n} & \binom{p+2n+1}{n} & \dots & \binom{p+3n}{n} \end{vmatrix}.$$

*384. $$\begin{vmatrix} 1 & \binom{p}{1} & \binom{p}{2} & \dots & \binom{p}{n} \\ 1 & \binom{p+1}{1} & \binom{p+1}{2} & \dots & \binom{p+1}{n} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{p+n}{1} & \binom{p+n}{2} & \dots & \binom{p+n}{n} \end{vmatrix}.$$

*385. $$\begin{vmatrix} \binom{m}{p} & \binom{m}{p+1} & \cdots & \binom{m}{p+n} \\ \binom{m+1}{p} & \binom{m+1}{p+1} & \cdots & \binom{m+1}{p+n} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{m+n}{p} & \binom{m+n}{p+1} & \cdots & \binom{m+n}{p+n} \end{vmatrix}.$$

*386. $$\begin{vmatrix} 1 & \binom{2}{2} & 0 & 0 & \cdots & 0 \\ 1 & \binom{3}{2} & \binom{3}{3} & 0 & \cdots & 0 \\ 1 & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \cdots & \binom{n}{n} \\ 1 & \binom{n+1}{2} & \binom{n+1}{4} & \binom{n+1}{4} & \cdots & \binom{n+1}{n} \end{vmatrix}.$$

*387. $$\begin{vmatrix} 2 & 3 & 4 & \cdots & n \\ 3 & 6 & 10 & \cdots & \frac{n(n+1)}{2!} \\ 4 & 10 & 20 & \cdots & \frac{n(n+1)(n+2)}{3!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n & \frac{n(n+1)}{2!} & \frac{n(n+1)(n+2)}{3!} & \cdots & \frac{n(n+1)\ldots(2n-2)}{(n-1)!} \end{vmatrix}.$$

*388. $$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & \binom{1}{1} & 0 & 0 & \dots & 0 & x \\ 1 & \binom{2}{1} & \binom{2}{2} & 0 & \dots & 0 & x^2 \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \dots & 0 & x^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-1} & x^n \end{vmatrix}.$$

*389. $$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 1! & 0 & 0 & 0 & \dots & x \\ 1 & 2 & 2! & 0 & 0 & \dots & x^2 \\ 1 & 3 & 3\cdot 2 & 3! & 0 & \dots & x^3 \\ 1 & 4 & 4\cdot 3 & 4\cdot 3\cdot 2 & 4! & \dots & x^4 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & n & n(n-1) & n(n-1)(n-2) & n(n-1)(n-2)(n-3) & \dots & x^n \end{vmatrix}.$$

*390. $$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & x_0 \\ 1 & \binom{1}{1} & 0 & 0 & \dots & 0 & x_1 \\ 1 & \binom{2}{1} & \binom{2}{2} & 0 & \dots & 0 & x_2 \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \dots & 0 & x_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-1} & x_n \end{vmatrix}.$$

*391. $$\begin{vmatrix} 0 & x & x & \dots & x \\ y & 0 & x & \dots & x \\ y & y & 0 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & 0 \end{vmatrix}.$$

392. $$\begin{vmatrix} a & x & x & \dots & x \\ y & a & x & \dots & x \\ y & y & a & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & a \end{vmatrix}.$$

393. $\begin{vmatrix} a_1 & x & x & \dots & x \\ y & a_2 & x & \dots & x \\ y & y & a_3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & a_n \end{vmatrix}$.

394. $\begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & a_1 & x & x & \dots & x \\ 1 & y & a_2 & x & \dots & x \\ 1 & y & y & a_3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & y & y & y & \dots & a_n \end{vmatrix}$.

395. $\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & 0 & \dots & 0 & 0 \\ 2 & \alpha+\beta & \alpha\beta & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix}$.

396. $\begin{vmatrix} a+1 & a & 0 & 0 & \dots & 0 & 0 \\ 1 & a+1 & a & 0 & \dots & 0 & 0 \\ 0 & 1 & a+1 & a & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & a+1 \end{vmatrix}$.

397. $\begin{vmatrix} n!a_0 & (n-1)!\,a_1 & (n-2)!\,a_2 & \dots & a_n \\ -n & x & 0 & \dots & 0 \\ 0 & -(n-1) & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix}$.

398. $\begin{vmatrix} \operatorname{cosec}\alpha & 1 & 0 & 0 & \dots & 0 \\ 1 & 2\operatorname{cosec}\alpha & 1 & 0 & \dots & 0 \\ 0 & 1 & 2\operatorname{cosec}\alpha & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2\operatorname{cosec}\alpha \end{vmatrix}$.

*399. $\begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & x & 2 & 0 & \dots & 0 & 0 \\ 0 & n-2 & x & 3 & \dots & 0 & 0 \\ 0 & 0 & n-3 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & x \end{vmatrix}$.

400.
$$\begin{vmatrix} a^p - x & a^{p+1} - x & \dots & a^{p+n-1} - x \\ a^{p+n} - x & a^{p+n+1} - x & \dots & a^{p+2n-1} - x \\ \dots & \dots & \dots & \dots \\ a^{p+n(n-1)} - x & a^{p+n(n-1)+1} - x & \dots & a^{p+n^2-1} - x \end{vmatrix}.$$

401.
$$\begin{vmatrix} 1-x & a & a^2 & \dots & a^{n-1} \\ a & a^2-x & a^3 & \dots & a^n \\ a^2 & a^3 & a^4-x & \dots & a^{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a^{n-1} & a^n & a^{n+1} & \dots & a^{2n-2}-x \end{vmatrix}.$$

402.
$$\begin{vmatrix} a_0 & 1 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & 0 & \dots & 0 \\ 1 & 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & a_n \end{vmatrix}.$$

403.
$$\begin{vmatrix} c_0 & b & b & b & \dots & b \\ a & c_1 & 0 & 0 & \dots & 0 \\ a & 0 & c_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a & 0 & 0 & 0 & \dots & c_n \end{vmatrix}.$$

***404.**
$$\begin{vmatrix} 1 & -b & -b & -b & \dots & -b \\ 1 & na & -2b & -3b & \dots & -(n-1)b \\ 1 & (n-1)a & a & -3b & \dots & -(n-1)b \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2a & a & a & \dots & a \end{vmatrix}.$$

***405.**
$$\begin{vmatrix} (x_1-a_1)^2 & a_2^2 & \dots & a_n^2 \\ a_1^2 & (x_2-a_2)^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^2 & a_2^2 & \dots & (x_n-a_n)^2 \end{vmatrix}.$$

406.
$$\begin{vmatrix} (x_1-a_1)^2 & a_1a_2 & a_1a_3 & \dots & a_1a_n \\ a_2a_1 & (x_2-a_2)^2 & a_2a_3 & \dots & a_2a_n \\ a_3a_1 & a_3a_2 & (x_3-a_3)^2 & \dots & a_3a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_na_1 & a_na_2 & a_na_3 & \dots & (x_n-a_n)^2 \end{vmatrix}.$$

***407.**
$$\begin{vmatrix} 1-b_1 & b_2 & 0 & 0 & \dots & 0 \\ -1 & 1-b_2 & b_3 & 0 & \dots & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1-b_n \end{vmatrix}.$$

408. $\begin{vmatrix} 0 & a_2 & a_3 & \dots & a_n \\ b_1 & 0 & a_3 & \dots & a_n \\ b_1 & b_2 & 0 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & \dots & 0 \end{vmatrix}$.

***409.** $\begin{vmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & 2 & 3 & \dots & n-1 \\ 1 & x & 1 & 2 & \dots & n-1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & x & \dots & 1 \end{vmatrix}$.

***410.** $\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ x & 1 & 2 & \dots & n-1 \\ x & x & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & 1 \end{vmatrix}$.

***411.** $\begin{vmatrix} a_0x^n & a_1x^{n-1} & a_2x^{n-2} & \dots & a_{n-1}x & a_n \\ a_0x & b_1 & 0 & \dots & 0 & 0 \\ a_0x^2 & a_1x & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0x^{n-1} & a_1x^{n-2} & a_2x^{n-3} & \dots & b_{n-1} & 0 \\ a_0x^n & a_1x^{n-1} & a_2x^{n-2} & \dots & a_{n-1}x & b_n \end{vmatrix}$.

412. $\begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+2 & x & \dots & x \\ x & x & x+3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & x+n \end{vmatrix}$.

413. $\begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+2 & x & \dots & x \\ x & x & x+4 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & x+2^n \end{vmatrix}$.

414. $\begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+\frac{1}{2} & x & \dots & x \\ x & x & x+\frac{1}{3} & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & x+\frac{1}{n} \end{vmatrix}$.

415. $\begin{vmatrix} x+1 & x & x & \dots & x \\ x & x+a & x & \dots & x \\ x & x & x+a^2 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & x+a^n \end{vmatrix}$.

***416.** $\begin{vmatrix} (a_1+b_1)^{-1} & (a_1+b_2)^{-1} & \dots & (a_1+b_n)^{-1} \\ \dots & \dots & \dots & \dots \\ (a_n+b_1)^{-1} & (a_n+b_2)^{-1} & \dots & (a_n+b_n)^{-1} \end{vmatrix}$.

***417.** $\begin{vmatrix} \frac{1}{x_1-a_1} & \frac{1}{x_1-a_2} & \dots & \frac{1}{x_1-a_n} \\ \dots & \dots & \dots & \dots \\ \frac{1}{x_n-a_1} & \frac{1}{x_n-a_2} & \dots & \frac{1}{x_n-a_n} \end{vmatrix}$.

***418.** $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \dots & \frac{1}{2n-1} \end{vmatrix}$.

***419.** $\begin{vmatrix} a_0+a_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & a_1+a_2 & a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & a_2+a_3 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & a_{n-1}+a_n \end{vmatrix}$.

***420.** Obtain the law for forming expanded expressions of nth-order continuants (the term "continuant" stems from the connection with continued fractions; see problem 539):

$$(a_1 a_2 \dots a_n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & a_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & a_3 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & a_n \end{vmatrix},$$

that is, an expression in the form of a polynomial in a_1, a_2, . . ., a_n. Write out in expanded form continuants of order 4, 5 and 6.

Sec. 6. Minors, Cofactors and the Laplace Theorem

421. How many minors of order k does an nth-order determinant have?

422. Prove that when determining the sign of a cofactor, one can use the sum of the numbers of the rows and columns of the complementary minor instead of the given minor. In other words, if M is the given minor, M' is the complementary minor, A is the cofactor of M, and A' is the cofactor of M', then from $A = \varepsilon M'$, where $\varepsilon = \pm 1$, it follows that $A' = \varepsilon M$.

423. Show that the Laplace expansion of an nth-order determinant in terms of any k rows (or columns) coincides with its expansion in terms of the remaining $n - k$ rows (or columns).

***424.** Show that the rule of signs connecting a cofactor A with the complementary minor M' of minor M can be stated thus: let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be row numbers, $\beta_1, \beta_2, \ldots, \beta_k$ column numbers of the minor M in a determinant D of order n written out in increasing order, and let

$$\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n \quad \text{and} \quad \beta_{k+1}, \beta_{k+2}, \ldots, \beta_n$$

be, respectively, the numbers of the rows and columns of the complementary minor M', also written out in the order of increasing magnitude; then $A = M'$ if the substitution $\begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_n \\ \beta_1, & \beta_2, & \ldots, & \beta_n \end{pmatrix}$ is even, and $A = -M'$ if the substitution is odd.

Using the Laplace theorem, evaluate the following determinants:

425. $\begin{vmatrix} 5 & 1 & 2 & 7 \\ 3 & 0 & 0 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 0 & 0 & 3 \end{vmatrix}$.

426. $\begin{vmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 0 & 8 \\ 3 & 0 & 0 & 2 \\ 4 & 4 & 7 & 5 \end{vmatrix}$.

427. $\begin{vmatrix} 0 & 5 & 2 & 0 \\ 8 & 3 & 5 & 4 \\ 7 & 2 & 4 & 1 \\ 0 & 4 & 1 & 0 \end{vmatrix}$.

428. $\begin{vmatrix} 5 & 2 & 1 & 3 & 2 \\ 4 & 0 & 7 & 0 & 0 \\ 2 & 3 & 7 & 5 & 3 \\ 2 & 3 & 6 & 4 & 5 \\ 3 & 0 & 4 & 0 & 0 \end{vmatrix}$.

429. $\begin{vmatrix} 2 & 1 & 4 & 3 & 5 \\ 3 & 4 & 0 & 5 & 0 \\ 3 & 4 & 5 & 2 & 1 \\ 1 & 5 & 2 & 4 & 3 \\ 4 & 6 & 0 & 7 & 0 \end{vmatrix}$.

430. $\begin{vmatrix} 7 & 2 & 1 & 3 & 4 \\ 1 & 0 & 2 & 0 & 3 \\ 3 & 0 & 4 & 0 & 7 \\ 6 & 3 & 2 & 4 & 5 \\ 5 & 1 & 2 & 2 & 3 \end{vmatrix}$.

431. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 0 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 3 & 5 & 2 & 4 \\ 0 & 5 & 0 & 3 & 2 \end{vmatrix}$.

432. $\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 7 & 6 & 5 & 4 & 0 & 0 \\ 2 & 3 & 4 & 5 & 0 & 0 \\ 5 & 1 & 2 & 6 & 7 & 3 \\ 2 & 7 & 5 & 3 & 4 & 1 \end{vmatrix}$.

433. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 3 \\ 6 & 5 & 7 & 8 & 4 & 2 \\ 9 & 8 & 6 & 7 & 0 & 0 \\ 3 & 2 & 4 & 5 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 & 0 \end{vmatrix}$.

434. $\begin{vmatrix} 7 & 6 & 5 & 4 & 3 & 2 \\ 9 & 7 & 8 & 9 & 4 & 3 \\ 7 & 4 & 9 & 7 & 0 & 0 \\ 5 & 3 & 6 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 6 & 8 & 0 & 0 \end{vmatrix}$.

435. $\begin{vmatrix} 2 & 3 & 0 & 0 & 1 & -1 \\ 9 & 4 & 0 & 0 & 3 & 7 \\ 4 & 5 & 1 & -1 & 2 & 4 \\ 3 & 8 & 3 & 7 & 6 & 9 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 0 \end{vmatrix}$.

436. $\begin{vmatrix} 1 & 0 & 2 & 0 & 3 & 0 \\ 5 & 1 & 4 & 2 & 7 & 3 \\ 1 & 0 & 4 & 0 & 9 & 0 \\ 8 & 1 & 5 & 3 & 7 & 6 \\ 1 & 0 & 8 & 0 & 27 & 0 \\ 9 & 1 & 5 & 4 & 3 & 10 \end{vmatrix}$.

437. $\begin{vmatrix} 1 & 1 & 0 & 0 \\ x_1 & x_2 & \cos\alpha & \sin\alpha \\ y_1 & y_2 & \cos\beta & \sin\beta \\ z_1 & z_2 & \cos\gamma & \sin\gamma \end{vmatrix}$.

438. $\begin{vmatrix} 0 & 0 & a & b & c \\ 0 & 0 & a' & b' & c' \\ a & a' & x_1 & y_3 & y_2 \\ b & b' & y_3 & x_2 & y_1 \\ c & c' & y_2 & y_1 & x_3 \end{vmatrix}$.

439. $\begin{vmatrix} 0 & a & b & c \\ 1 & x & 0 & 0 \\ 1 & 0 & y & 0 \\ 1 & 0 & 0 & z \end{vmatrix}$.

440. $\begin{vmatrix} 0 & a & b & c \\ a' & 1 & 0 & 0 \\ b' & 0 & 1 & 0 \\ c' & 0 & 0 & 1 \end{vmatrix}$.

441. $\begin{vmatrix} 1 & x & x & x \\ 1 & a & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{vmatrix}$.

442. $\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix}$.

443.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,2n-2} & a_{1,2n-1} & a_{1,2n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,2n-2} & a_{2,2n-1} & 0 \\ 0 & 0 & a_{33} & \dots & a_{3,2n-2} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & a_{2n-2,3} & \dots & a_{2n-2,2n-2} & 0 & 0 \\ 0 & a_{2n-1,2} & a_{2n-1,3} & \dots & a_{2n-1,2n-2} & a_{2n-1,2n-1} & 0 \\ a_{2n,1} & a_{2n,2} & a_{2n,3} & \dots & a_{2n,2n-2} & a_{2n,2n-1} & a_{2n,2n} \end{vmatrix}.$$

444.

$$\begin{vmatrix} a_{11} & 1 & a_{12} & 1 & \dots & a_{1n} & 1 \\ 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ a_{21} & x_1 & a_{22} & x_2 & \dots & a_{2n} & x_n \\ x_1 & 0 & x_2 & 0 & \dots & x_n & 0 \\ a_{31} & x_1^2 & a_{32} & x_2^2 & \dots & a_{3n} & x_n^2 \\ x_1^2 & 0 & x_2^2 & 0 & \dots & x_n^2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & x_1^{n-1} & a_{n2} & x_2^{n-1} & \dots & a_{nn} & x_n^{n-1} \\ x_1^{n-1} & 0 & x_2^{n-1} & 0 & \dots & x_n^{n-1} & 0 \end{vmatrix}.$$

Using the Laplace theorem, compute the following determinants after first transforming them:

*445. $\begin{vmatrix} 2 & -1 & 3 & 4 & -5 \\ 4 & -2 & 7 & 8 & -7 \\ -6 & 4 & -9 & -2 & 3 \\ 3 & -2 & 4 & 1 & -2 \\ -2 & 6 & 5 & 4 & -3 \end{vmatrix}$.

446. $\begin{vmatrix} 5 & -5 & -3 & 4 & 2 \\ -4 & 4 & 3 & 6 & 3 \\ 3 & -1 & 5 & -9 & -5 \\ -7 & 7 & 6 & 8 & 4 \\ 5 & -3 & 2 & -1 & -2 \end{vmatrix}$.

447. $\begin{vmatrix} 2 & 1 & 2 & 3 & 2 \\ 3 & -2 & 7 & 5 & -1 \\ 3 & -1 & -5 & -3 & -2 \\ 5 & -6 & 4 & 2 & -4 \\ 2 & -3 & 3 & 1 & -2 \end{vmatrix}$.

448. $\begin{vmatrix} 2 & -3 & 5 & -2 & 1 \\ 3 & 2 & 5 & -4 & -3 \\ -2 & 3 & -4 & 2 & -3 \\ 6 & 4 & 7 & -8 & -1 \\ 2 & -1 & 7 & 1 & 5 \end{vmatrix}$.

449. $\begin{vmatrix} 5 & 9 & -2 & -4 & 5 \\ 2 & -3 & 4 & -3 & 3 \\ -5 & -7 & 2 & 4 & -2 \\ 4 & -5 & 8 & -6 & 8 \\ 6 & -5 & 3 & -3 & 7 \end{vmatrix}$.

450.
$$\begin{vmatrix} 3 & 4 & -3 & -1 & 2 \\ -5 & 6 & 5 & 2 & 3 \\ 4 & -9 & -3 & 7 & -5 \\ -1 & -4 & 1 & 1 & -2 \\ -3 & 7 & 5 & 2 & 3 \end{vmatrix}.$$

451.† $$\begin{vmatrix} 1+x & x & \dots & x & x & \dots & x & 1+x \\ x & 1+x & \dots & x & x & \dots & 1+x & x \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & 1+x & 1+x & \dots & x & x \\ x & x & \dots & 1+2x & 1+x & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x & 1+2x & \dots & x & x & \dots & 1+x & x \\ 1+2x & x & \dots & x & x & \dots & x & 1+x \end{vmatrix}.$$

452.† $$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1n} & 0 & 0 & \dots & 0 & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2,n-1} & 0 & 0 & 0 & \dots & b_{2,n-1} & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & 0 & 0 & b_{n1} & b_{n2} & \dots & b_{n,n-1} & b_{nn} \\ c_{11} & c_{12} & \dots & c_{1,n-1} & c_{1n} & 0 & 0 & \dots & 0 & b_{1n} \\ c_{21} & c_{22} & \dots & c_{2,n-1} & 0 & 0 & 0 & \dots & b_{2,n-1} & b_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n1} & 0 & \dots & 0 & 0 & b_{n1} & b_{n2} & \dots & b_{n,n-1} & b_{nn} \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 & x & a_1 & a_2-1 & \dots & a_{n-1}-1 & a_n-1 \\ 1 & 1 & \dots & x & 1 & a_1-1 & a_2 & \dots & a_{n-1}-1 & a_n-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x & 1 & \dots & 1 & 1 & a_1-1 & a_2-1 & \dots & a_{n-1}-1 & a_n \\ a_1-x & a_1 & \dots & a_1 & a_1 & -a_1 & -a_1 & \dots & -a_1 & x-a_1 \\ a_2 & a_2-x & \dots & a_2 & a_2 & -a_2 & -a_2 & \dots & x-a_2 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & a_n & \dots & a_n & a_n-x & x-a_n & -a_n & \dots & -a_n & -a_n \end{vmatrix}.$$

† The order of the determinant is equal to $2n$.

454. Isolate, in a determinant D of even order $n = 2k$, four minors M_1, M_2, M_3, M_4 of order k, as shown below:

$$\left|\begin{array}{c|c} \underset{M_1}{\begin{matrix} a_{11} & \dots & a_{1k} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{matrix}} & \underset{M_2}{\begin{matrix} a_{1,\,k+1} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{k,\,k+1} & \dots & a_{k,\,n} \end{matrix}} \\ \hline \underset{M_3}{\begin{matrix} a_{k+1,\,1} & \dots & a_{k+1,\,k} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nk} \end{matrix}} & \underset{M_4}{\begin{matrix} a_{k+1,\,k+1} & \dots & a_{k+1,\,n} \\ \dots & \dots & \dots \\ a_{n,\,k+1} & \dots & a_{nn} \end{matrix}} \end{array}\right|.$$

Express D in terms of M_1, M_2, M_3, M_4 in the following two cases:

(a) if all elements of M_2 or M_3 are zero;

(b) if all elements of M_1 or M_4 are zero.

455. In a determinant D of order $n = kl$, let there be isolated l kth-order minors on the secondary diagonal; that is, M_1 lies in the first k rows and the last k columns, M_2 in the next k rows and the preceding k columns, and so forth, and, finally, M_l lies in the last k rows and the first k columns.

Express D in terms of M_1, M_2, . . ., M_l if all elements of D on one side of the indicated chain of minors are equal to zero.

456. In a determinant D of order n, let there be isolated k rows and l columns, $l \leqslant k$, and let all elements of the isolated l columns not lying in the isolated k rows be zero. Show that in the Laplace expansion of D in terms of the isolated k rows, one must only take those kth-order minors which contain the isolated l columns; this assertion holds true if the rows and columns are interchanged.

457. Use the Laplace theorem to solve problem 206.

458. Prove that

$$\begin{vmatrix} a_{11} & 0 & a_{12} & 0 & \dots & a_{1n} & 0 \\ 0 & b_{11} & 0 & b_{12} & \dots & 0 & b_{1n} \\ a_{21} & 0 & a_{22} & 0 & \dots & a_{2n} & 0 \\ 0 & b_{21} & 0 & b_{22} & \dots & 0 & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & 0 & a_{n2} & 0 & \dots & a_{nn} & 0 \\ 0 & b_{n1} & 0 & b_{n2} & \dots & 0 & b_{nn} \end{vmatrix}.$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}.$$

*459. Compute the following determinant of order $k + l$:

$$\left.\begin{vmatrix} 3 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 3 & 2 & 0 & \dots & 0 & \end{vmatrix}\right\} k \text{ rows.}$$

$$\left.\begin{vmatrix} 0 & \dots & 0 & 2 & 5 & 3 & \dots & 0 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 & 5 \end{vmatrix}\right\} l \text{ rows.}$$

460. Write out the expansion of the following nth-order continuant (compare with problem 420):

$$(a_1, a_2, \dots, a_n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & a_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & a_3 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & a_n \end{vmatrix}$$

in terms of the first k rows. What is the property of the Fibonacci numbers (see problem 365) here when $n = 2k$?

461. Prove, without removing parentheses, that the equation

$$(ab' - a'b)(cd' - c'd) - (ac' - a'c)(bd' - b'd) + (ad' - a'd)(bc' - b'c) = 0$$

holds true for arbitrary values of a, b, c, d, a', b', c', d'.

*462. In the matrix

$$\left(\begin{array}{ccc|ccc} a_{11} & \dots & a_{1n} & a_{1,\,n+1} & \dots & a_{1,\,2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & a_{n,\,n+1} & \dots & a_{n,\,2n} \end{array}\right),$$

containing n rows and $2n$ columns, take any minor M of order n containing at least half the columns of the left half of the matrix.

Let σ be the sum of the numbers of the columns of minor M and let M' be a minor of order n made up of the remaining columns of the matrix. Prove that $\sum (-1)^{\sigma} MM' = 0$, where the sum is taken over all minors M of the indicated type.

***463.** Show that the determinants

$$D = \begin{vmatrix} a_1x_1 & b_1x_1 & a_1x_2 & b_1x_2 & a_1x_3 & b_1x_3 \\ a_2x_1 & b_2x_1 & a_2x_2 & b_2x_2 & a_2x_3 & b_2x_3 \\ a_1y_1 & b_1y_1 & a_1y_2 & b_1y_2 & a_1y_3 & b_1y_3 \\ a_2y_1 & b_2y_1 & a_2y_2 & b_2y_2 & a_2y_3 & b_2y_3 \\ a_1z_1 & b_1z_1 & a_1z_2 & b_1z_2 & a_1z_3 & b_1z_3 \\ a_2z_1 & b_2z_1 & a_2z_2 & b_2z_2 & a_2z_3 & b_2z_3 \end{vmatrix},$$

$$\delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ and } \Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

are related by the equality $D = \delta^3\Delta^2$ (this property is generalized in problem 540).

***464.** Suppose

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$
$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4,$$
$$h(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

and

$$(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px^2 + qx + r.$$

Show that

$$\begin{vmatrix} f(\alpha) & f(\beta) & f(\gamma) \\ g(\alpha) & g(\beta) & g(\gamma) \\ h(\alpha) & h(\beta) & h(\gamma) \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} \cdot \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ r & q & p & 1 & 0 \\ 0 & r & q & p & 1 \end{vmatrix}.$$

465. We say the determinant

$$D=\begin{vmatrix} a_{11} & \dots & a_{1n} & x_{11} & \dots & x_{1k} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & x_{n1} & \dots & x_{nk} \\ y_{11} & \dots & y_{1n} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_{k1} & \dots & y_{kn} & 0 & \dots & 0 \end{vmatrix}$$

is formed by bordering the determinant

$$\Delta=\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

with k rows and k columns.

Show that when $k > n$, $D = 0$, and when $k \leqslant n$, D is a form (that is, a homogeneous polynomial) of degree $n - k$ in the elements a_{ij} of the determinant Δ and a form of degree $2k$ in the bordering elements x_{ij}, y_{ij} whose coefficients are the cofactors of kth-order minors in the determinant Δ. Namely, prove that $D = (-1)^k \sum A_{i_1 i_2 \dots i_k j_1 j_2 \dots j_k} \times \times X_{i_1 i_2 \dots i_k} Y_{j_1 j_2 \dots j_k}$, where $A_{i_1 i_2 \dots i_k j_1 j_2 \dots j_k}$ is the cofactor of the minor of the determinant Δ in the rows labelled $i_1, i_2, \dots, i_k$ and in the columns labelled $j_1, j_2, \dots, j_k$, while $X_{i_1 i_2 \dots i_k}$ and $Y_{j_1 j_2 \dots j_k}$ are minors of D made up of bordering elements and lying in the rows (and columns) with the indicated labels. Here the sum is taken over all combinations of indices from 1 to n with $i_1 < i_2 < \dots < i_k$, $j_1 < j_2 < \dots < j_k$.

*466. Prove the following generalization of the Laplace theorem. Given an nth-order determinant D: D is equal to the sum of all possible products of the type indicated below if the rows of D are split into p systems without common rows, the first system including rows with labels $\alpha_1 < \alpha_2 < \dots < \alpha_k$, the second, rows with labels $\alpha_{k+1} < \alpha_{k+2} < \dots < \alpha_{k+l}$, etc., and, finally, the last system, rows with labels $\alpha_{n-s+1} < \alpha_{n-s+2} < \dots < \alpha_n$; if in the matrix of the first system of rows we take a minor M_1 of order k lying in columns with labels $\beta_1 < \beta_2 < \dots < \beta_k$, in the second matrix a minor M_2 of order l lying in columns with labels $\beta_{k+1} < \beta_{k+2} < \dots < \beta_{k+l}$ that differ from the labels of the columns of M_1, and so on, and, finally, in the last ma-

trix we take a minor M_p of order s lying in the remaining columns with labels $\beta_{n-s+1} < \beta_{n-s+2} < \ldots < \beta_n$; and if we then form the product $\varepsilon M_1 M_2 \ldots M_p$, where $\varepsilon = +1$ if the substitution

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} \tag{1}$$

is even, and $\varepsilon = -1$ if that substitution is odd. That this assertion is a generalization of the Laplace theorem follows from problem 424.

Sec. 7. Multiplication of Determinants

467. Multiply the determinants

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 4 & 5 & 4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & -3 & 1 \\ 1 & -4 & 3 \\ 1 & -5 & 2 \end{vmatrix}$$

in all four possible ways (by multiplying the rows or columns of the first determinant by the rows or columns of the second one) and verify that in all cases the value of the resulting determinant is equal to the product of the values of the given determinants.

468. Compute the following determinant by squaring it:

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}.$$

469. Compute the following determinant by squaring it

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ -b & a & d & -c & f & -e & -h & g \\ -c & -d & a & b & g & h & -e & -f \\ -d & c & -b & a & h & -g & f & -e \\ -e & -f & -g & -h & a & b & c & d \\ -f & e & -h & g & -b & a & -d & c \\ -g & h & e & -f & -c & d & a & -b \\ -h & -g & f & e & -d & -c & b & a \end{vmatrix}.$$

Evaluate the following determinants by representing them as products of determinants:

*470. $$\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \dots & \dots & \dots & \dots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}.$$

471. $$\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \dots & \cos(\alpha_1-\beta_n) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \dots & \cos(\alpha_2-\beta_n) \\ \dots & \dots & \dots & \dots \\ \cos(\alpha_n-\beta_1) & \cos(\alpha_n-\beta_2) & \dots & \cos(\alpha_n-\beta_n) \end{vmatrix}.$$

472. $$\begin{vmatrix} 1 & \cos(\alpha_1-\alpha_2) & \cos(\alpha_1-\alpha_3) & \dots & \cos(\alpha_1-\alpha_n) \\ \cos(\alpha_1-\alpha_2) & 1 & \cos(\alpha_2-\alpha_3) & \dots & \cos(\alpha_2-\alpha_n) \\ \cos(\alpha_1-\alpha_3) & \cos(\alpha_2-\alpha_3) & 1 & \dots & \cos(\alpha_3-\alpha_n) \\ \dots & \dots & \dots & \dots & \dots \\ \cos(\alpha_1-\alpha_n) & \cos(\alpha_2-\alpha_n) & \cos(\alpha_3-\alpha_n) & \dots & 1 \end{vmatrix}.$$

473. $$\begin{vmatrix} \sin 2\alpha_1 & \sin(\alpha_1+\alpha_2) & \dots & \sin(\alpha_1+\alpha_n) \\ \sin(\alpha_2+\alpha_1) & \sin 2\alpha_2 & \dots & \sin(\alpha_2+\alpha_n) \\ \dots & \dots & \dots & \dots \\ \sin(\alpha_n+\alpha_1) & \sin(\alpha_n+\alpha_2) & \dots & \sin 2\alpha_n \end{vmatrix}.$$

474. $$\begin{vmatrix} \frac{1-a_1^n b_1^n}{1-a_1b_1} & \frac{1-a_1^n b_2^n}{1-a_1b_2} & \dots & \frac{1-a_1^n b_n^n}{1-a_1b_n} \\ \frac{1-a_2^n b_1^n}{1-a_2b_1} & \frac{1-a_2^n b_2^n}{1-a_2b_2} & \dots & \frac{1-a_2^n b_n^n}{1-a_2b_n} \\ \dots & \dots & \dots & \dots \\ \frac{1-a_n^n b_1^n}{1-a_nb_1} & \frac{1-a_n^n b_2^n}{1-a_nb_2} & \dots & \frac{1-a_n^n b_n^n}{1-a_nb_n} \end{vmatrix}.$$

475. $$\begin{vmatrix} (a_0+b_0)^n & (a_0+b_1)^n & \dots & (a_0+b_n)^n \\ (a_1+b_0)^n & (a_1+b_1)^n & \dots & (a_1+b_n)^n \\ \dots & \dots & \dots & \dots \\ (a_n+b_0)^n & (a_n+b_1)^n & \dots & (a_n+b_n)^n \end{vmatrix}.$$

*476. $$\begin{vmatrix} 1^{n-1} & 2^{n-1} & \dots & n^{n-1} \\ 2^{n-1} & 3^{n-1} & \dots & (n+1)^{n-1} \\ \dots & \dots & \dots & \dots \\ n^{n-1} & (n+1)^{n-1} & \dots & (2n-1)^{n-1} \end{vmatrix}.$$

477. $\begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ s_2 & s_3 & s_4 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} \end{vmatrix}$, where $s_k = x_1^k + x_2^k + \dots + x_n^k$.

*478. $\begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & s_3 & \dots & s_n & x \\ s_2 & s_3 & s_4 & \dots & s_{n+1} & x^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n-1} & x^n \end{vmatrix}$, where $s_k = x_1^k + x_2^k + \dots + x_n^k$.

*479. Prove that the value of the circulant is given by the equality

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{vmatrix} = f(\varepsilon_1) f(\varepsilon_2) \dots f(\varepsilon_n),$$

where $f(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^{n-1}$ and ε_1, ε_2, ..., ε_n are all values of the nth root of unity.

480. Prove, using the notation of problem 479, that

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & a_4 & \dots & a_1 \\ a_3 & a_4 & a_5 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_1 & a_2 & \dots & a_{n-1} \end{vmatrix} = (-1)^{\frac{(n-1)(n-2)}{2}} f(\varepsilon_1) f(\varepsilon_2) \dots f(\varepsilon_n).$$

*481. Compute the determinant

$$\begin{vmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ \alpha^{n-1} & 1 & \alpha & \dots & \alpha^{n-2} \\ \alpha^{n-2} & \alpha^{n-1} & 1 & \dots & \alpha^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha & \alpha^2 & \alpha^3 & \dots & 1 \end{vmatrix}.$$

482. Compute the following determinant, using the result of problem 479:

$$\begin{vmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \dots & \dots & \dots & \dots \\ b & b & \dots & a \end{vmatrix}.$$

483. Use the result of problem 479 to compute the determinant

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}.$$

Evaluate the following determinants:

484. $$\begin{vmatrix} 1 & C_n^1 & C_n^2 & \dots & C_n^{n-1} & 1 \\ 1 & 1 & C_n^1 & \dots & C_n^{n-2} & C_n^{n-1} \\ C_n^{n-1} & 1 & 1 & \dots & C_n^{n-3} & C_n^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_n^1 & C_n^2 & C_n^3 & \dots & 1 & 1 \end{vmatrix}.$$

485. $$\begin{vmatrix} 1 & 2a & 3a^2 & \dots & na^{n-1} \\ na^{n-1} & 1 & 2a & \dots & (n-1)a^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 2a & 3a^2 & 4a^3 & \dots & 1 \end{vmatrix}.$$

***486.** Prove that

$$\begin{vmatrix} s-a_1 & s-a_2 & \dots & s-a_n \\ s-a_n & s-a_1 & \dots & s-a_{n-1} \\ \dots & \dots & \dots & \dots \\ s-a_2 & s-a_3 & \dots & s-a_1 \end{vmatrix} = (-1)^{n-1}(n-1) \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \dots & \dots & \dots & \dots \\ a_2 & a_3 & \dots & a_1 \end{vmatrix},$$

where $s = a_1 + a_2 + \dots + a_n$.

Compute the following determinants:

*487. $\begin{vmatrix} \overbrace{-1 & -1 & -1 & \dots & -1}^{(p \text{ columns})} & \overbrace{1 & 1 & \dots & 1 & 1}^{(n-p \text{ columns})} \\ 1 & -1 & -1 & \dots & -1 & -1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & 1 & 1 & 1 & \dots & 1 & -1 \end{vmatrix}$.

*488. $\begin{vmatrix} \overbrace{a & a & a & \dots & a}^{(p \text{ columns})} & \overbrace{b & b & \dots & b & b}^{(n-p \text{ columns})} \\ b & a & a & \dots & a & a & b & \dots & b & b \\ b & b & a & \dots & a & a & a & \dots & b & b \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & b & b & b & \dots & b & a \end{vmatrix}$.

*489. $\begin{vmatrix} \cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \cos\frac{3\pi}{n} & \dots & -1 \\ -1 & \cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \dots & \cos\frac{(n-1)\pi}{n} \\ \cos\frac{(n-1)\pi}{n} & -1 & \cos\frac{\pi}{n} & \dots & \cos\frac{(n-2)\pi}{n} \\ \dots & \dots & \dots & \dots & \dots \\ \cos\frac{2\pi}{n} & \cos\frac{3\pi}{n} & \cos\frac{4\pi}{n} & \dots & \cos\frac{\pi}{n} \end{vmatrix}$.

*490. $\begin{vmatrix} \cos\theta & \cos 2\theta & \cos 3\theta & \dots & \cos n\theta \\ \cos n\theta & \cos\theta & \cos 2\theta & \dots & \cos(n-1)\theta \\ \dots & \dots & \dots & \dots & \dots \\ \cos 2\theta & \cos 3\theta & \cos 4\theta & \dots & \cos\theta \end{vmatrix}$.

491. $\begin{vmatrix} \sin a & \sin(a+h) & \sin(a+2h) & \dots & \sin[a+(n-1)h] \\ \sin[a+(n-1)h] & \sin a & \sin(a+h) & \dots & \sin[a+(n-2)h] \\ \dots & \dots & \dots & \dots & \dots \\ \sin(a+h) & \sin(a+2h) & \sin(a+3h) & \dots & \sin a \end{vmatrix}$.

*492. $\begin{vmatrix} 1^2 & 2^2 & 3^2 & \dots & n^2 \\ n^2 & 1^2 & 2^2 & \dots & (n-1)^2 \\ (n-1)^2 & n^2 & 1^2 & \dots & (n-2)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 2^2 & 3^2 & 4^2 & \dots & 1^2 \end{vmatrix}$.

*493. Compute the following skew circulant (or skew-cyclic determinant):

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{vmatrix}.$$

494.

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n z & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} z & a_n z & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_2 z & a_3 z & a_4 z & \dots & a_1 \end{vmatrix},$$ where z is any number.

*495. Prove that a circulant of order $2n$ with the first row made up of the elements $a_1, a_2, \dots, a_{2n-1}, a_{2n}$ is equal to the product of a circulant of order n with the first row made up of elements $a_1 + a_{n+1}, a_2 + a_{n+2}, \dots, a_n + a_{2n}$ and a skew circulant of order n with the first row made up of the elements $a_1 - a_{n+1}, a_2 - a_{n+2}, \dots, a_n - a_{2n}$.

*496. Multiply the two determinants

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x & x_2 & -x_1 \end{vmatrix} \cdot \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{vmatrix}$$

to prove the Euler identity:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2)$$
$$= (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2$$
$$+ (x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2)^2 + (x_1y_4 - x_2y_4 + x_3y_2 - x_4y_1)^2.$$

What property of the integers follows from this?

*497. Use multiplication of determinants to prove the identity

$$(a^3 + b^3 + c^3 - 3abc)(a'^3 + b'^3 + c'^3 - 3a'b'c')$$
$$= A^3 + B^3 + C^3 - 3ABC,$$

where $A = aa' + bc' + cb'$, $B = ac' + bb' + ca'$, $C = ab' + ba' + cc'$. What property of the integers follows from this?

*498. Using the notation of problem 497, prove the following identity:

$$(a^2 + b^2 + c^2 - ab - ac - bc) \times (a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c') = A^2 + B^2 + C^2 - AB - AC - BC.$$

*499. Prove the following generalization of the theorem on multiplication of determinants: given two matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix},$$

each having m rows and n columns.

Combining the rows of one matrix with the rows of the other, putting $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{jk}$, set up an mth-order determinant:

$$D = \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mm} \end{vmatrix}.$$

Denote by $A_{i_1, i_2, \dots, i_m}$ and $B_{i_1, i_2, \dots, i_m}$ respectively the mth-order minors of matrices A and B made up of the columns (of these matrices) labelled $i_1, i_2, \dots, i_m$ in the same order. Then

$$D = \sum_{1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n} A_{i_1, i_2, \dots, i_m} B_{i_1, i_2, \dots, i_m} \qquad (1)$$

for $m \leqslant n$ (the Binet-Cauchy formula), that is, the determinant D is equal to the sum of the products of all mth-order minors of matrix A into the corresponding minors of matrix B. When $m > n$,

$$D = 0. \qquad (2)$$

*500. Prove assertion (2) of the preceding problem by using the theorem on the multiplication of determinants.

***501.** Prove the following Cauchy identity without carrying out the multiplication:

$$(a_1c_1+a_2c_2+\ldots+a_nc_n)(b_1d_1+b_2d_2+\ldots+b_nd_n)$$
$$-(a_1d_1+a_2d_2+\ldots+a_nd_n)(b_1c_1+b_2c_2+\ldots+b_nc_n)$$
$$=\sum_{1\leqslant i<k\leqslant n}(a_ib_k-a_kb_i)(c_id_k-c_kd_i)\quad(n>1).$$

502. Prove the Lagrange identity without multiplying:

$$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-\left(\sum_{i=1}^{n}a_ib_i\right)^2=\sum_{1\leqslant i<k\leqslant n}(a_ib_k-a_kb_i)^2.$$

***503.** Prove that for any real numbers

$$a_1,\ a_2,\ \ldots,\ a_n\quad\text{and}\quad b_1,\ b_2,\ \ldots,\ b_n$$

the following inequality holds true:

$$(a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2)$$
$$\geqslant(a_1b_1+a_2b_2+\ldots+a_nb_n)^2.$$

Note that the equality holds if and only if one of the given sets of numbers differs from the other solely by a numerical factor (which may be zero). (The Cauchy-Bunyakovsky inequality.)

***504.** Prove that for any complex numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ the following equality holds:

$$\left(\sum_{k=1}^{n}a_k\bar{a}_k\right)\left(\sum_{k=1}^{n}b_k\bar{b}_k\right)-\left(\sum_{k=1}^{n}a_k\bar{b}_k\right)\left(\sum_{k=1}^{n}\bar{a}_kb_k\right)$$
$$=\sum_{1\leqslant j<k\leqslant n}(a_jb_k-a_kb_j)(\bar{a}_j\bar{b}_k-\bar{a}_k\bar{b}_j).$$

***505.** Prove that for any two sets of complex numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ the following inequality holds true:

$$\left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right)\geqslant\left|\sum_{k=1}^{n}a_k\bar{b}_k\right|^2.$$

Note that the equality sign holds if and only if the numbers of one of the given sets differ from those of the other by a numerical factor alone.

***506.** The *adjugate determinant* of a determinant D of order $n>1$ is a determinant D' obtained from D by re-

placing all the elements with their cofactors (the location of the elements is retained).

Prove that

$$D' = D^{n-1}. \tag{1}$$

*507. Let M be an mth-order minor fo a determinant D, let A be the cofactor of M, let M' be a minor of the adjugate determinant D' corresponding to the minor M (that is, made up of the cofactors of the elements of D that enter into M). Prove the equality $M' = D^{m-1}A$.

If we agree to regard the complementary minor of the entire determinant D as equal to 1, then this equality will be a generalization of the equality of problem 506 (for $m = n$).

*508. Let C be a minor of the $(n-2)$th order obtained from the determinant D by crossing out the ith and jth rows and the kth and lth columns with $i < j$ and $k < l$; A_{pq} is, as usual, the cofactor of the element a_{pq}. Prove that

$$\begin{vmatrix} A_{ik} & A_{il} \\ A_{jk} & A_{jl} \end{vmatrix} = (-1)^{i+j+k+l} DC.$$

*509. Show that if the determinant D is equal to zero then all the rows (and also the columns) of the adjugate determinant are proportional.

*510. Let a_{ij} be an element of determinant D of order n and let A'_{ij} be the cofactor of the corresponding element A_{ij} of determinant D', the adjugate of D. Show that $A'_{ij} = D^{n-2}a_{ij}$.

*511. Let M be a minor of order m of the determinant D of order n, and let M' be the minor, corresponding to M, of the adjugate determinant D' and let A' be the cofactor of the minor M'. Prove that $A' = D^{n-m-1}M$. This is a generalization of the equality of problem 510.

*512. Knowing the minors of all elements of a nonzero determinant D, find the elements.

*513. Let

$$s_k = x_1^k + x_2^k + \ldots + x_n^k \quad (k = 1, 2, 3, \ldots),$$

$$p = x_1 x_2 \ldots x_n.$$

Show that

$$\begin{vmatrix} n+1 & s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & s_3 & \dots & s_{n+1} \\ s_2 & s_3 & s_4 & \dots & s_{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n} \end{vmatrix} = p^2 \begin{vmatrix} n & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ s_2 & s_3 & s_4 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} \end{vmatrix}.$$

514. Show that if

$$D(x) = \begin{vmatrix} a_{11}-x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-x & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn}-x \end{vmatrix},$$

then the product $D(x) \cdot D(-x)$ can be expressed as

$$\begin{vmatrix} A_{11}-x^2 & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22}-x^2 & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn}-x^2 \end{vmatrix},$$

where all the A_{ij} are independent of x. Find the expression of A_{ij} in terms of a_{kl}.

***515.** Use multiplication of determinants to prove that a determinant reverses sign under an interchange of two rows (or two columns).

***516.** Use multiplication of determinants to prove that a determinant remains unchanged if a row (or column) multiplied by a number c is added to another row (or column).

***517.** Show that the determinant

$$\begin{vmatrix} 1 & \cos\varphi_3 & \cos\varphi_2 \\ \cos\varphi_3 & 1 & \cos\varphi_1 \\ \cos\varphi_2 & \cos\varphi_1 & 1 \end{vmatrix}$$ is zero if $\varphi_1 + \varphi_2 + \varphi_3 = 0$.

***518.** Let l_1, l_2, l_3 and m_1, m_2, m_3 be the cosines of the angles of two rays with the orthogonal coordinate axes and let φ be the angle between the rays. Prove that $\sin^2\varphi = (l_1m_2 - l_2m_1)^2 + (l_2m_3 - l_3m_2)^2 + (l_3m_1 - l_1m_3)^2$.

519. Let α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3 be the angles of three rays L_1, L_2, L_3 with the orthogonal coordinate axes and let the angles between the rays be $\varphi_1 = \angle(L_2, L_3)$,

$\varphi_2 = \angle (L_3, L_1)$, $\varphi_3 = \angle (L_1, L_2)$. Prove that

$$\begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{vmatrix}^2$$
$$= 1 - \cos^2\varphi_1 - \cos^2\varphi_2 - \cos^2\varphi_3 + 2\cos\varphi_1\cos\varphi_2\cos\varphi_3.$$

*520. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the rectangular coordinates of the points M_1, M_2, M_3 in the plane. Show that the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

remains unchanged under a rotation of the coordinate axes and a translation of the origin. Use this fact to elucidate the geometric meaning of the determinant.

*521. Let (x_1, y_1) and (x_2, y_2) be the rectangular coordinates of two points M_1 and M_2 in the plane. After determining the geometric meaning of the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$, figure out whether it undergoes change under a rotation of the axes and a translation of the coordinate origin.

*522. Compute the product of the determinants

$$\begin{vmatrix} x_1 & y_1 & R \\ x_2 & y_2 & R \\ x_3 & y_3 & R \end{vmatrix} \cdot \begin{vmatrix} -x_1 & -y_1 & R \\ -x_2 & -y_2 & R \\ -x_3 & -y_3 & R \end{vmatrix}$$

and thus obtain an expression of the radius of a circumscribed circle in terms of the sides a, b, c and the area S of a triangle.

*523. Let $\alpha_1, \alpha_2, \alpha_3$; $\beta_1, \beta_2, \beta_3$; $\gamma_1, \gamma_2, \gamma_3$ be, respectively, the cosines of the angles of three pairwise orthogonal rays OA, OB, OC with axes Ox, Oy, Oz of a rectangular coordinate system. Prove that the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \pm 1;$$

the plus sign appears for the same orientation of the trihedrons $OABC$ and $Oxyz$ (this means that when the figure $OABC$ is rotated, OA and Ox, OB and

Oy, OC and Oz can be brought to coincidence), the minus sign appears for opposite orientation (this means that when OA and Ox, and OB and Oy are brought to coincidence, the rays OC and Oz are in opposite directions).

*524. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the coordinates of three points M_1, M_2, M_3 of space. Show that the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

remains unchanged under a rotation of the coordinate system (which is assumed to be rectangular) and determine the geometric meaning of the determinant.

*525. Find an expression for the volume V of a parallelepiped in terms of the lengths a, b, c of its edges passing through a single vertex and the angles α, β, γ formed by the edges. (The angle α is formed by b and c; β by c and a; and γ by a and b.)

*526. Let l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3 be the cosines of the angles of rays OA, OB, OC with the positive semi-axes Ox, Oy, Oz of a rectangular coordinate system.

Prove that for the rays OA, OB and OC to be coplanar, it is necessary and sufficient that

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0.$$

*527. Let (x_i, y_i, z_i) be the rectangular coordinates of a point M_i of space ($i = 1, 2, 3, 4$). Determine the geometric meaning of the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z & 1 \end{vmatrix}$$

by showing that it does not change under a translation of the origin.

*528. Multiply together the determinants

$$\begin{vmatrix} x_1 & y_1 & z_1 & R \\ x_2 & y_2 & z_2 & R \\ x_3 & y_3 & z_3 & R \\ x_4 & y_4 & z_4 & R \end{vmatrix} \text{ and } \begin{vmatrix} -x_1 & -y_1 & -z_1 & R \\ -x_2 & -y_2 & -z_2 & R \\ -x_3 & -y_3 & -z_3 & R \\ -x_4 & -y_4 & -z_4 & R \end{vmatrix}$$

to obtain an expression for the radius of a sphere circumscribed about an arbitrary tetrahedron in terms of the volume and an edge of the tetrahedron. In particular, use the expression obtained to find the radius of a sphere circumscribed about a regular tetrahedron with the length of an edge a.

Sec. 8. Miscellaneous Problems

*529. Show that a determinant of order n permits of the following axiomatic definition (which is equivalent to the ordinary definition).

Any row of n numbers (or of the elements of any field P) will be called a vector and will be denoted by a single letter in boldface type. The addition of two vectors and the multiplication of a vector by a scalar (number) are defined as usual, that is, if

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad \boldsymbol{b} = (b_1, b_2, \ldots, b_n)$$

then

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),$$

if c is a scalar, then $c\boldsymbol{a} = (ca_1, ca_2, \ldots, ca_n)$.

A function $f(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ of n vectors with numerical values is termed a linear function in each argument (or, simply, polylinear) if

$$f(\boldsymbol{a}_1, \ldots, c'\boldsymbol{a}_i' + c''\boldsymbol{a}_i'', \ldots, \boldsymbol{a}_n)$$

$$= c'f(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i', \ldots, \boldsymbol{a}_n) + c''f(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i'', \ldots, \boldsymbol{a}_n) \quad (\alpha)$$

for any vectors, any numbers c', c'' and any $i = 1, 2, \ldots, n$. Also we say that a function has the property of annihilation if

$$f(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_j, \ldots, \boldsymbol{a}_n) = 0 \quad \text{for} \quad \boldsymbol{a}_i = \boldsymbol{a}_j \quad (\beta)$$

$$i, j = 1, 2, \ldots, n, \quad i \neq j.$$

Let $\boldsymbol{e}_i$ $(i = 1, 2, \ldots, n)$ be a vector with unity in the ith position and zeros elsewhere. The function $f(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots,$

$\ldots, a_n)$ is said to be normalized if

$$f(e_1, e_2, \ldots, e_n) = 1 \qquad (\gamma)$$

Given a square matrix of order n,

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

and, in the ordinary sense, its determinant $|A|$, that is,

$$|A| = \sum (-1)^s a_{1i_1} a_{2i_2} \ldots a_{ni_n},$$

where the sum is taken over all permutations $i_1, i_2, \ldots, i_n$ of the numbers $1, 2, \ldots, n$, and s is the number of inversions in each permutation.

Show that

(1) the determinant $|A|$, as a function of the rows of matrix A, has the properties (α), (β), (γ);

(2) any function of n vectors having the properties (α) and (β) satisfies the equality $f(a_1, a_2, \ldots, a_n) = |A| f(e_1, e_2, \ldots, e_n)$, where A is a matrix with rows $a_1, a_2, \ldots, a_n$;

(3) any function $f(a_1, a_2, \ldots, a_n)$ having the properties (α), (β), (γ) is equal to the determinant $|A|$ of matrix A with the rows $a_1, a_2, \ldots, a_n$. In other words, the determinant $|A|$ of matrix A is the sole polylinear, normalized function of its rows with the property of annihilation.

*530. Use assertion (2) of the preceding problem to prove the theorem of the multiplication of determinants.

531. Show that for functions of n vectors over a field with characteristic different from 2, the property (β), given property (α), is equivalent to the alternating-sign property of the function, that is,

$$f(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = -f(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n) \qquad (\beta')$$

for arbitrary vectors and arbitrary $i, j = 1, 2, \ldots, n$, $i \neq j$. Construct an example of a function of n vectors over a field P with characteristic 2 having the properties (α), (β') and (γ), but not having the property (β).

*532. Compute the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^6 & \dots & \varepsilon^{2(n-1)} \\ 1 & \varepsilon^3 & \varepsilon^6 & \varepsilon^9 & \dots & \varepsilon^{3(n-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \varepsilon^{3(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{vmatrix},$$

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

*533. What changes does a determinant undergo if we isolate k rows (or columns) and subtract from each one the remaining isolated rows?

534. Express the determinant

$$D = \begin{vmatrix} a_{11}+x & a_{12}+x & \dots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \dots & a_{2n}+x \\ \dots & \dots & \dots & \dots \\ a_{n1}+x & a_{n2}+x & \dots & a_{nn}+x \end{vmatrix}$$

as a polynomial arranged in powers of x.

*535. Prove that the sum of the cofactors of all elements of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

is equal to the determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_{21}-a_{11} & a_{22}-a_{12} & \dots & a_{2n}-a_{1n} \\ a_{31}-a_{11} & a_{32}-a_{12} & \dots & a_{3n}-a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \dots & a_{nn}-a_{1n} \end{vmatrix}.$$

*536. Prove that the sum of the cofactors of all elements of a determinant remains unchanged if the same number is added to all elements.

*537. Prove that if all elements of some row (or column) of a determinant are equal to unity, then the sum of the cofactors of all elements of the determinant is equal to the determinant itself.

538. Prove that a skew-symmetric determinant of even order remains unchanged if the same number is added to all elements.

*539. Establish the following relationship of continuants (see problem 420 for an expanded expression of a continuant)

$$(a_1 a_2 \ldots a_n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & a_2 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & a_3 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & -1 & a_n \end{vmatrix}$$

and continued fractions

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n} = \frac{(a_1 a_2 \ldots a_n)}{(a_2 a_3 \ldots a_n)}.$$

*540. Given two determinants:

$$A = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \text{ of order } n$$

and

$$B = \begin{vmatrix} b_{11} & b_{12} & \ldots & b_{1p} \\ b_{21} & b_{22} & \ldots & b_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ b_{p1} & b_{p2} & \ldots & b_{pp} \end{vmatrix} \text{ of order } p.$$

Form a determinant of order np:

$$D=\begin{vmatrix}
a_{11}b_{11} & \dots & a_{1n}b_{11} & a_{11}b_{12} & \dots & a_{1n}b_{12} & \dots & a_{11}b_{1p} & \dots & a_{1n}b_{1p} \\
a_{21}b_{11} & \dots & a_{2n}b_{11} & a_{21}b_{12} & \dots & a_{2n}b_{12} & \dots & a_{21}b_{1p} & \dots & a_{2n}b_{1p} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{n1}b_{11} & \dots & a_{nn}b_{11} & a_{n1}b_{12} & \dots & a_{nn}b_{12} & \dots & a_{n1}b_{1p} & \dots & a_{nn}b_{1p} \\
a_{11}b_{21} & \dots & a_{1n}b_{21} & a_{11}b_{22} & \dots & a_{1n}b_{22} & \dots & a_{11}b_{2p} & \dots & a_{1n}b_{2p} \\
a_{21}b_{21} & \dots & a_{2n}b_{21} & a_{21}b_{22} & \dots & a_{2n}b_{22} & \dots & a_{21}b_{2p} & \dots & a_{2n}b_{2p} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{n1}b_{21} & \dots & a_{nn}b_{21} & a_{n1}b_{22} & \dots & a_{nn}b_{22} & \dots & a_{n1}b_{2p} & \dots & a_{nn}b_{2p} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{n1}b_{p1} & \dots & a_{nn}b_{p1} & a_{n1}b_{p2} & \dots & a_{nn}b_{p2} & \dots & a_{n1}b_{pp} & \dots & a_{nn}b_{pp}
\end{vmatrix}.$$

Thus the matrix of the determinant D consists of p^2 blocks with n rows and n columns in each. Here, the block in the ith block row and the jth block column (for arbitrary $i, j = 1, 2, \dots, p$) is obtained from the matrix of determinant A by multiplying all its elements by b_{ij}. Prove that $D = A^pB^n$. The determinant D is said to be the Kronecker product of the determinants A and B (see problems 963, 965).

541. Prove the following rule for expanding a bordered determinant: if

$$D=\begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
\dots & \dots & \dots & \dots \\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{vmatrix}$$

and A_{ij} is the cofactor of element a_{ij}, then

$$\begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1n} & x_1 \\
a_{21} & a_{22} & \dots & a_{2n} & x_2 \\
\dots & \dots & \dots & \dots & \dots \\
a_{n1} & a_{n2} & \dots & a_{nn} & x_n \\
y_1 & y_2 & \dots & y_n & z
\end{vmatrix} = Dz - \sum_{i,j=1}^{n} A_{ij}x_iy_j.$$

***542.** Suppose the elements of a determinant D are polynomials of the unknowns $x_1, x_2, \dots, x_s$ with numerical coefficients (or coefficients of an arbitrary field P), and $D = 0$. Prove that it is possible to represent the cofactors

of the elements of D in the form $A_{ij} = A_i B_j$, $i, j = 1, 2, \ldots$ $\ldots, n$, where all the A_i and B_j are polynomials of $x_1, x_2, \ldots, x_s$. Find these polynomials for the determinant Δ:

$$\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix},$$

where the unknowns are a, b, c.

***543.** Use the two preceding problems to prove that a skew-symmetric determinant of even order is the square of some polynomial of its elements that lie above the principal diagonal.

***544.** Show that if in the general expression of a skew-symmetric determinant every element a_{ji} for $j > i$ is replaced by $-a_{ij}$, then all terms will cancel whose substitutions of the indices (when decomposed into cycles) yield at least one cycle of odd length.

***545.** Let D be a skew-symmetric determinant of even order n with elements $a_{ij} = -a_{ji}$ $(i, j = 1, 2, \ldots, n)$. The *Pfaffian product of determinant* D is the product

$$\varepsilon a_{i_1, i_2} a_{i_3, i_4} \ldots a_{i_{n-1}, i_n},$$

in which the indices of $n/2$ component elements form a permutation $i_1, i_2, \ldots, i_n$ of the numbers $1, 2, \ldots, n$; $\varepsilon = +1$ if the permutation is even and $\varepsilon = -1$ if the permutation is odd. The Pfaffian product is said to be *reduced* if it consists solely of elements above the principal diagonal in D (that is, if the first index of each element is less than the second index). We will use the term *essential* to describe a term of determinant D if the substitution of its indices has cycles of even length only. A pair of reduced Pfaffian products N_1, N_2 (in a given order) is said to *correspond* to a given essential term of D if it is constructed on the basis of that term in the following manner. Suppose a substitution of indices of a given term is written cyclically thus:

$$(\alpha_1 \alpha_2 \ldots \alpha_{h-1} \alpha_h)\,(\beta_1 \beta_2 \ldots \beta_{g-1} \beta_g) \ldots (\mu_1 \mu_2 \ldots \mu_{k-1} \mu_k) \quad (1)$$

Here, $\alpha_1 = 1$ and each cycle, from the second onwards, begins with the smallest number of numbers that do not appear in the preceding cycles. We now construct the Pfaffian products

$$N_1' = \varepsilon_1 a_{\alpha_1, \alpha_2} a_{\alpha_3, \alpha_4} \ldots a_{\alpha_{h-1}, \alpha_h} a_{\beta_1, \beta_2} a_{\beta_3, \beta_4} \ldots a_{\beta_{g-1}, \beta_g} \ldots a_{\mu_1, \mu_2} a_{\mu_3, \mu_4} \ldots a_{\mu_{k-1}, \mu_k}$$

and

$$N_2' = \varepsilon_2 a_{\alpha_2, \alpha_3} a_{\alpha_4, \alpha_5} \ldots a_{\alpha_k, \alpha_1} a_{\beta_2, \beta_3} a_{\beta_4, \beta_5} \ldots a_{\beta_g, \beta_1} \ldots a_{\mu_2, \mu_3} a_{\mu_4, \mu_5} \ldots a_{\mu_k, \mu_1},$$

and then each element a_{ij}, where $i > j$, is replaced by $-a_{ji}$.

Accordingly, the sign of ε_1 or ε_2 is changed, but the substitution class is also changed so that for each replacement we again obtain a Pfaffian product. Thus, by carrying out in N_1' and N_2' all the indicated replacements, we obtain the pair N_1, N_2 of reduced Pfaffian products that corresponds to the given essential term of D.

Prove that

(1) any pair of reduced Pfaffian products (distinct or identical) corresponds to one and only one essential term of the general expansion of determinant D. (In the general expansion of D, terms obtained from one another by replacements of the type $a_{ij} = -a_{ji}$, are regarded as distinct.) In other words, a one-to-one correspondence has been established between all essential terms and all pairs of reduced Pfaffian products of the determinant D;

(2) each essential term is equal to the product of reduced Pfaffian products of the corresponding pair;

(3) $D = p^2$, where p is the sum of all reduced Pfaffian products; it is called the Pfaffian aggregate, or the Pfaffian, of the determinant D.

***546.** Prove the following recurrence formula that is convenient for computing the Pfaffian defined in the preceding problem. If p_n is the Pfaffian of the skew-symmetric determinant $D_n = |a_{ij}|$ of even order $n > 2$, and p_{in} is the Pfaffian of the determinant D_{in} obtained from D_n by crossing out the nth and ith rows, and also the corresponding columns, where $i = 1, 2, \ldots, n-1$, then

$$p_n = \sum_{i=1}^{n-1} (-1)^{i-1} p_{in} a_{in}; \quad p_2 = a_{12}.$$

Show that p_{in} results from p_{n-2} by increasing by unity all indices of the elements greater than or equal to i.

547. Use the formula of the preceding problem to compute the Pfaffians p_2, p_4, p_6.

548. Use the formula of problem 546 to find the number of addends of the Pfaffian p_n of the skew-symmetric deter-

minant D_n of even order n, that is, the number of distinct reduced Pfaffian products of D_n (products that differ solely in the order of the factors are not regarded as distinct).

***549.** Let D be a skew-symmetric determinant of odd order n with elements a_{ij} $(i, j = 1, 2, \ldots, n)$. M_{ij} is the minor of the element a_{ij}, $p_{i,\ n+1}$ is the Pfaffian of the minor M_{ii}. Show that $M_{ij} = p_{i,\ n+1}p_{j,\ n+1}$ $(i, j = 1, 2, \ldots \ldots, n)$. Also show that for the polynomials A_i, B_j of problem 542 we can take $A_i = (-1)^{i-1}\, p_{i,\ n+1}$, $B_j = = (-1)^{j-1}\, p_{j,\ n+1}$ (provided that the unknowns $x_1, \ldots, x_s$ are elements of D lying above the principal diagonal).

Verify that in this manner we obtain for the determinant

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

the same result as in problem 542 (if one takes into account that in problem 542 A_i and B_j are determined to within change of sign in all the polynomials).

***550.** Prove that a determinant of general form, regarded as a polynomial of its elements assumed to be the unknowns, cannot be decomposed into two factors, each of which is a polynomial in the same unknowns of a nonzero degree. In other words, the determinant is an irreducible polynomial of its elements and, what is more, over any field.

***551.** Let $D = |\ a_{ij}\ |$ be a determinant of order $n > 1$, and let k be any one of the numbers $1, 2, \ldots, n$, $\binom{n}{k} = = C_n^k = \frac{n!}{k!\,(n-k)!}$. Denote by $s_1, s_2, \ldots, s_{\binom{n}{k}}$ all possible combinations of n numbers $1, 2, \ldots, n$, taken k at a time, that are numbered in an arbitrary (but, from then on, invariable) order. For the sake of definiteness, the numbers in each combination can be assumed to be arranged in the order of increasing magnitude, although this is not essential for what follows. Let μ_{ij} be the kth-order minor of determinant D at the intersection of rows with labels taken from the combination s_i and columns with labels taken from the combination s_j, where $i, j = 1, 2, \ldots, \binom{n}{k}$. Let α_{ij} be the cofactor of the minor μ_{ij} in D. We use the term determinant of minors of kth order of the determi-

nant D for a determinant of order $\binom{n}{k}$ having the form

$$\Delta_k = \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1\binom{n}{k}} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2\binom{n}{k}} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{\binom{n}{k}1} & \mu_{\binom{n}{k}2} & \cdots & \mu_{\binom{n}{k}\binom{n}{k}} \end{vmatrix}.$$

We introduce yet another determinant, $\overline{\Delta}_k$ of order $\binom{n}{k}$, that is obtained from Δ_k by replacing each minor μ_{ij} by its cofactor α_{ij} in D.

Prove that

(1) the values of the determinants Δ_k and $\overline{\Delta}_k$ do not change under a change in the numbering of the combinations, that is, under a permutation of the combinations in the sequence $s_1, s_2, \ldots, s_{\binom{n}{k}}$;

(2) $\Delta_k = \overline{\Delta}_{n-k}$. This is a generalization of the assertion of problem 242;

(3) $\Delta_k \overline{\Delta}_k = D^{\binom{n}{k}}$; (4) $\Delta_k = D^{\binom{n-1}{k-1}}$; (5) $\overline{\Delta}_k = D^{\binom{n-1}{k}}$.

***552.** Compute the determinant $P_n = |p_{ij}|$ in which $p_{ij} = 1$ if i divides j, and $p_{ij} = 0$ if i does not divide j. Find the value of the determinant $Q_n = |q_{ij}|$ in which q_{ij} is equal to the number of common divisors of the numbers i and j.

***553.** *Euler's function* is the function $\varphi(n)$ equal to the number of integers in the sequence $1, 2, \ldots, n$ that are prime to n. Using the preceding problem and the Gauss theorem which states that $n = \sum \varphi(d)$, where the sum is taken over all divisors d of the number n (including 1 and n itself), show that the nth-order determinant $D = |d_{ij}|$, where d_{ij} is the largest common divisor of the numbers i and j, is equal to $\varphi(1)\,\varphi(2) \ldots \varphi(n)$.

Chapter II

Systems of Linear Equations

Sec. 9. Systems of Equations Solved by the Cramer Rule

Solve the following systems of equations by Cramer's rule:

554. $2x_1 + 2x_2 - x_3 + x_4 = 4,$
$4x_1 + 3x_2 - x_3 + 2x_4 = 6,$
$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12,$
$3x_1 + 3x_2 - 2x_3 + 2x_4 = 6.$

555. $2x_1 + 3x_2 + 11x_3 + 5x_4 = 2,$
$x_1 + x_2 + 5x_3 + 2x_4 = 1,$
$2x_1 + x_2 + 3x_3 + 2x_4 = -3,$
$x_1 + x_2 + 3x_3 + 4x_4 = -3.$

556. $2x_1 + 5x_2 + 4x_3 + x_4 = 20,$
$x_1 + 3x_2 + 2x_3 + x_4 = 11,$
$2x_1 + 10x_2 + 9x_3 + 7x_4 = 40,$
$3x_1 + 8x_2 + 9x_3 + 2x_4 = 37.$

557. $3x_1 + 4x_2 + x_3 + 2x_4 + 3 = 0,$
$3x_1 + 5x_2 + 3x_3 + 5x_4 + 6 = 0,$
$6x_1 + 8x_2 + x_3 + 5x_4 + 8 = 0,$
$3x_1 + 5x_2 + 3x_3 + 7x_4 + 8 = 0.$

558. $7x_1 + 9x_2 + 4x_3 + 2x_4 - 2 = 0,$
$2x_1 - 2x_2 + x_3 + x_4 - 6 = 0,$
$5x_1 + 6x_2 + 3x_3 + 2x_4 - 3 = 0,$
$2x_1 + 3x_2 + x_3 + x_4 = 0.$

559. $6x + 5y - 2z + 4t + 4 = 0,$
$9x - y + 4z - t - 13 = 0,$
$3x + 4y + 2z - 2t - 1 = 0,$
$3x - 9y + 2t - 11 = 0.$

560. $\begin{aligned} 2x - y - 6z + 3t + 1 &= 0, \\ 7x - 4y + 2z - 15t + 32 &= 0, \\ x - 2y - 4z + 9t - 5 &= 0, \\ x - y + 2z - 6t + 8 &= 0. \end{aligned}$

561. $\begin{aligned} 2x + y + 4z + 8t &= -1, \\ x + 3y - 6z + 2t &= 3, \\ 3x - 2y + 2z - 2t &= 8, \\ 2x - y + 2z &= 4. \end{aligned}$

562. $\begin{aligned} 2x - y + 3z &= 9, \\ 3x - 5y + z &= -4, \\ 4x - 7y + z &= 5. \end{aligned}$

563. $\begin{aligned} 2x - 5y + 3z + t &= 5, \\ 3x - 7y + 3z - t &= -1, \\ 5x - 9y + 6z + 2t &= 7, \\ 4x - 6y + 3z + t &= 8. \end{aligned}$

*564. Two systems of linear equations with the same unknowns (though not necessarily having the same number of equations) are said to be equivalent if any solution of one system satisfies the other system. (Any two systems with the same unknowns, each system having no solution, are also regarded as equivalent.)

Show that any one of the following transformations of a system of linear equations carries a given system into an equivalent system:

(a) interchanging two equations;

(b) multiplying both sides of one of the equations by any nonzero number;

(c) termwise subtraction of one equation multiplied by any number from another.

Does a change in the numbering of the unknowns carry a given system into an equivalent system? Is it permissible to alter the numbering of the unknowns when solving a system of equations?

565. Prove that any system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, s \tag{1}$$

can, via transformations of type (a), (b), (c) of the preceding problem and via a change in the numbering of the unknowns, be reduced to the form

$$\sum_{j=1}^{n} c_{ij}y_j = d_i, \quad i = 1, 2, \ldots, s \tag{2}$$

which satisfies one and only one of the following three groups of conditions:

(a) $c_{ii} \neq 0$, $i = 1, 2, \ldots, n$; $c_{ij} = 0$ for $i > j$ (in particular, the coefficients of the unknowns in all equations beyond the nth—for $s > n$—are zero), $d_i = 0$ for $i = n + 1, \ldots, s$ (in this case we say that the system has been reduced to triangular form);

(b) there exists an integer r, $0 \leqslant r \leqslant n - 1$ such that $c_{ii} \neq 0$, $i = 1, 2, \ldots, r$; $c_{ij} = 0$ for $i > j$; $c_{ij} = 0$ for $i > r$ and any j equal to $1, 2, \ldots, n$; $d_i = 0$ for $i = r + 1, r + 2, \ldots, s$;

(c) there exists an integer r, $0 \leqslant r \leqslant n$ such that $c_{ii} \neq 0$ for $i = 1, 2, \ldots, r$; $c_{ij} = 0$ for $i > j$; $c_{ij} = 0$ for $i > r$ and any $j = 1, 2, \ldots, n$. There exists an integer k, $r + 1 \leqslant k \leqslant s$ such that $d_k \neq 0$.

Show that if in system (2) we restore the original numbering of the unknowns, the result will be a system equivalent to the original system (1). Then show that in case (a) the system (2) (and, hence, (1) as well) has a unique solution; in case (b), the system (2) has an infinity of solutions, wherein for any values of the unknowns $y_{r+1}, \ldots, y_n$ there exists a unique system of values of the remaining unknowns $y_1, \ldots, y_r$; in case (c) the system (2) has no solutions. This theorem justifies the method of elimination of unknowns when solving systems of linear equations.

566. Show that if the system of linear equations (1) of the preceding problem has integer coefficients, then under all transformations involved in reducing it to the form (2) fractions can be avoided so that system (2) will have integer coefficients.

Solve the following systems of equations by the elimination method:

567.
$$\begin{aligned} 3x_1 - 2x_2 - 5x_3 + x_4 &= 3, \\ 2x_1 - 3x_2 + x_3 + 5x_4 &= -3, \\ x_1 + 2x_2 - 4x_4 &= -3, \\ x_1 - x_2 - 4x_3 + 9x_4 &= 22. \end{aligned}$$

568.
$$\begin{aligned} 4x_1 - 3x_2 + x_3 + 5x_4 - 7 &= 0, \\ x_1 - 2x_2 - 2x_3 - 3x_4 - 3 &= 0, \\ 3x_1 - x_2 + 2x_3 + 1 &= 0, \\ 2x_1 + 3x_2 + 2x_3 - 8x_4 + 7 &= 0. \end{aligned}$$

569. $2x_1 - 2x_2 + x_4 + 3 = 0,$
$2x_1 + 3x_2 + x_3 - 3x_4 + 6 = 0,$
$3x_1 + 4x_2 - x_3 + 2x_4 = 0,$
$x_1 + 3x_2 + x_3 - x_4 - 2 = 0.$

570. $x_1 + x_2 - 6x_3 - 4x_4 = 6,$
$3x_1 - x_2 - 6x_3 - 4x_4 = 2,$
$2x_1 + 3x_2 + 9x_3 + 2x_4 = 6,$
$3x_1 + 2x_2 + 3x_3 + 8x_4 = -7.$

571. $2x_1 - 3x_2 + 3x_3 + 2x_4 - 3 = 0,$
$6x_1 + 9x_2 - 2x_3 - x_4 + 4 = 0,$
$10x_1 + 3x_2 - 3x_3 - 2x_4 - 3 = 0,$
$8x_1 + 6x_2 + x_3 + 3x_4 + 7 = 0.$

572. $x_1 + 2x_2 + 5x_3 + 9x_4 = 79,$
$3x_1 + 13x_2 + 18x_3 + 30x_4 = 263,$
$2x_1 + 4x_2 + 11x_3 + 16x_4 = 146,$
$x_1 + 9x_2 + 9x_3 + 9x_4 = 92.$

573. $x_1 + x_2 + x_3 + x_4 + x_5 = 15,$
$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 35,$
$x_1 + 3x_2 + 6x_3 + 10x_4 + 15x_5 = 70,$
$x_1 + 4x_2 + 10x_3 + 20x_4 + 35x_5 = 126,$
$x_1 + 5x_2 + 15x_3 + 35x_4 + 70x_5 = 210.$

574. $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 2,$
$2x_1 + 3x_2 + 7x_3 + 10x_4 + 13x_5 = 12,$
$3x_1 + 5x_2 + 11x_3 + 16x_4 + 21x_5 = 17,$
$2x_1 - 7x_2 + 7x_3 + 7x_4 + 2x_5 = 57,$
$x_1 + 4x_2 + 5x_3 + 3x_4 + 10x_5 = 7.$

***575.** $6x_1 + 6x_2 + 5x_3 + 18x_4 + 20x_5 = 14,$
$10x_1 + 9x_2 + 7x_3 + 24x_4 + 30x_5 = 18,$
$12x_1 + 12x_2 + 13x_3 + 27x_4 + 35x_5 = 32,$
$8x_1 + 6x_2 + 6x_3 + 15x_4 + 20x_5 = 16,$
$4x_1 + 5x_2 + 4x_3 + 15x_4 + 15x_5 = 11.$

576. $x_1 + x_2 + 4x_3 + 4x_4 + 9x_5 + 9 = 0,$
$2x_1 + 2x_2 + 17x_3 + 17x_4 + 82x_5 + 146 = 0,$

$$\begin{aligned} 2x_1 \quad &+ 3x_3 - x_4 + 4x_5 + 10 = 0, \\ x_2 + 4x_3 &+ 12x_4 + 27x_5 + 26 = 0, \\ x_1 + 2x_2 + 2x_3 &+ 10x_4 \quad - 37 = 0. \end{aligned}$$

577. $$\begin{aligned} 5x_1 + 2x_2 - 7x_3 + 14x_4 &= 21, \\ 5x_1 - x_2 + 8x_3 - 13x_4 + 3x_5 &= 12, \\ 10x_1 + x_3 - 2x_3 + 7x_4 - x_5 &= 29, \\ 15x_1 + 3x_2 + 15x_3 + 9x_4 + 7x_5 &= 130, \\ 2x_1 - x_2 - 4x_3 + 5x_4 - 7x_5 &= -13. \end{aligned}$$

578. $$\begin{aligned} 2x_1 + 7x_2 + 3x_3 + x_4 &= 5, \\ x_1 + 3x_2 + 5x_3 - 2x_4 &= 3, \\ x_1 + 5x_2 - 9x_3 + 8x_4 &= 1, \\ 5x_1 + 18x_2 + 4x_3 + 5x_4 &= 12. \end{aligned}$$

579. $$\begin{aligned} 2x_1 + 3x_2 - x_3 + x_4 &= 1, \\ 8x_1 + 12x_2 - 9x_3 + 8x_4 &= 3, \\ 4x_1 + 6x_2 + 3x_3 - 2x_4 &= 3, \\ 2x_1 + 3x_2 + 9x_3 - 7x_4 &= 3. \end{aligned}$$

580. $$\begin{aligned} 4x_1 - 3x_2 + 2x_3 - x_4 &= 8, \\ 3x_1 - 2x_2 + x_3 - 3x_4 &= 7, \\ 2x_1 - x_2 \quad - 5x_4 &= 6, \\ 5x_1 - 3x_2 + x_3 - 8x_4 &= 1. \end{aligned}$$

581. $$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 3, \\ 4x_1 - 2x_2 - 2x_3 + 3x_4 &= 2, \\ 2x_1 - x_2 + 5x_3 - 6x_4 &= 1, \\ 2x_1 - x_2 - 3x_3 + 4x_4 &= 5. \end{aligned}$$

582. Show that a polynomial of degree n is fully determined by its values for $n + 1$ values of the unknown. To be more precise, show that for any distinct numbers $x_0, x_1, x_2, \ldots, x_n$ and any numbers $y_0, y_1, \ldots, y_n$ there exists one, and only one, polynomial $f(x)$ of degree $\leqslant n$ for which

$$f(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n.$$

583. Using the preceding problem, prove the equivalence of two definitions of the equality of polynomials of one unknown (induction can be used to prove with ease a similar

assertion for polynomials of any number of unknowns) with numerical coefficients (or coefficients of any infinite field):

(1) two polynomials are said to be equal if the coefficients of each pair of terms of the same degree are equal (this is the accepted definition in algebra);

(2) two polynomials are said to be equal if they are equal as functions, that is, if the values for each value of the unknowns are equal (this is the accepted definition in analysis).

584. Show that the definitions given in the preceding problem are not equivalent for a finite field of coefficients (construct an example)

585. Find the quadratic polynomial $f(x)$ if we know that

$$f(1) = -1; \quad f(-1) = 9; \quad f(2) = -3.$$

586. Find a polynomial of degree three, $f(x)$, for which

$$f(-1) = 0, \quad f(1) = 4, \quad f(2) = 3, \quad f(3) = 16.$$

587. What does the assertion of problem 582 mean geometrically?

588. Find a parabola of degree 3 passing through the points (0, 1), (1, −1), (2, 5), (3, 37), the asymptotic direction being parallel to the axis of ordinates.

589. Find a fourth-degree parabola that passes through the points (5, 0), (−13, 2), (−10, 3), (−2, 1), (14, −1), the asymptotic direction being parallel to the axis of abscissas.

Solve the following systems of linear equations, using the most suitable device in each case:

590.
$$\begin{aligned} -x + y + z + t &= a, \\ x - y + z + t &= b, \\ x + y - z + t &= c, \\ x + y + z - t &= d. \end{aligned}$$

***591.**
$$\begin{aligned} a(x + t) + b(y + z) &= c, \\ a'(y + t) + b'(z + x) &= c', \\ a''(z + t) + b''(x + y) &= c'', \\ x + y + z + t &= d, \end{aligned}$$

where $a \neq b$, $a' \neq b'$, $a'' \neq b''$.

***592.**
$$\begin{aligned} ax + by + cz + dt &= p, \\ -bx + ay + dz - ct &= q, \end{aligned}$$

$$-cx - dy + az + bt = r,$$
$$-dx + cy - bz + at = s.$$

***593.**
$$x_n + a_1 x_{n-1} + a_1^2 x_{n-2} + \ldots + a_1^{n-1} x_1 + a_1^n = 0,$$
$$x_n + a_2 x_{n-1} + a_2^2 x_{n-2} + \ldots + a_2^{n-1} x_1 + a_2^n = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n + a_n x_{n-1} + a_n^2 x_{n-2} + \ldots + a_n^{n-1} x_1 + a_n^n = 0,$$

where $a_1, a_2, \ldots, a_n$ are distinct numbers.

594.
$$x_1 + x_2 + \ldots + x_n = 1,$$
$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b,$$
$$a_1^2 x_1 + a_2^2 x_2 + \ldots + a_n^2 x_n = b^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_1^{n-1} x_1 + a_2^{n-1} x_2 + \ldots + a_n^{n-1} x_n = b^{n-1},$$

where $a_1, a_2, \ldots, a_n$ are distinct numbers.

595.
$$x_1 + a_1 x_2 + \ldots + a_1^{n-1} x_n = b_1,$$
$$x_1 + a_2 x_2 + \ldots + a_2^{n-1} x_n = b_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_1 + a_n x_2 + \ldots + a_n^{n-1} x_n = b_n,$$

where $a_1, a_2, \ldots, a_n$ are distinct numbers.

596.
$$x_1 + x_2 + \ldots + x_n = b_1,$$
$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_1^{n-1} x_1 + a_2^{n-1} x_2 + \ldots + a_n^{n-1} x_n = b_n,$$

where $a_1, a_2, \ldots, a_n$ are distinct numbers.

597.
$$x_1 + x_2 + \ldots + x_n + 1 = 0,$$
$$2x_1 + 2^2 x_2 + \ldots + 2^n x_n + 1 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$nx_1 + n^2 x_2 + \ldots + n^n x_n + 1 = 0.$$

598.
$$ax_1 + bx_2 + \ldots + bx_n = c_1,$$
$$bx_1 + ax_2 + \ldots + bx_n = c_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$bx_1 + bx_2 + \ldots + ax_n = c_n,$$

where $(a - b)\,[a + (n - 1)\,b] \neq 0$.

*599. $(3+2a_1)x_1+(3+2a_2)x_2+\ldots+(3+2a_n)x_n$

$$=3+2b,$$

$$(1+3a_1+2a_1^2)x_1+(1+3a_2+2a_2^2)x_2$$

$$\ldots+(1+3a_n+2a_n^2)x_n=1+3b+2b^2,$$

$$a_1(1+3a_1+2a_1^2)x_1+a_2(1+3a_2+2a_2^2)x_2$$

$$\ldots+a_n(1+3a_n+2a_n^2)x_n=b(1+3b+2b^2),$$

. . . . . .

$$a_1^{n-3}(1+3a_1+2a_1^2)x_1+a_2^{n-3}(1+3a_2+2a_2^2)x_2$$

$$\ldots+a_n^{n-3}(1+3a_n+2a_n^2)x_n=b^{n-3}(1+3b+2b^2),$$

$$a_1^{n-2}(1+3a_1)x_1+a_2^{n-2}(1+3a_2)x_2$$

$$\ldots+a_n^{n-2}(1+3a_n)x_n=b^{n-2}(1+3b).$$

*600. Expanding the function $\frac{x}{\ln(1+x)}$ in a power series, we obtain $\frac{x}{\ln(1+x)}=1+h_1x+h_2x^2+h_3x^3+\ldots$.

Show that

$$h_n=\begin{vmatrix} \frac{1}{2} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \ldots & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n+1} & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \ldots & \frac{1}{2} \end{vmatrix}.$$

601. It will be recalled that $\frac{1}{\cos x}=1+\frac{e_1}{2!}x^2+\frac{e_2}{4!}x^4+$ $+\frac{e_3}{6!}x^6+\ldots$, where e_1, e_2, e_3 are the so-called Euler

numbers. Show that

$$e_n=(2n)!\begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \frac{1}{(2n-6)!} & \cdots & \frac{1}{2!} \end{vmatrix}.$$

*602. In the expansion $\frac{x}{e^x-1}=1+b_1x+b_2x^2+b_3x^3+\dots$, $b_{2n}=\frac{(-1)^{n-1}B_n}{(2n)!}$, where the B_n are the so-called Bernoulli numbers. Show that

$$B_n=(-1)^{n+1}(2n)!\begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2n+1)!} & \frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{2!} \end{vmatrix}.$$

Also show that

$$b_{2n-1}=\begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-3)!} & \cdots & \frac{1}{2!} \end{vmatrix}=0$$

for $n>1$.

*603. Show that the Bernoulli numbers B_n that were introduced in the preceding problem can be expressed by

the following nth-order determinants:

$$B_n = \frac{1}{2}(2n)! \begin{vmatrix} \frac{1}{3!} & 1 & 0 & 0 & \dots & 0 \\ \frac{3}{5!} & \frac{1}{3!} & 1 & 0 & \dots & 0 \\ \frac{5}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{2n-1}{(2n+1)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \frac{1}{(2n-5)!} & \dots & \frac{1}{3!} \end{vmatrix}$$

or

$$B_n = 2^n (2n)! \begin{vmatrix} \frac{1}{4!} & \frac{1}{2!} & 0 & 0 & \dots & 0 \\ \frac{2}{6!} & \frac{1}{4!} & \frac{1}{2!} & 0 & \dots & 0 \\ \frac{3}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{n}{(2n+2)!} & \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \dots & \frac{1}{4!} \end{vmatrix}.$$

*604. Denote by $s_n(k)$ the sum of the nth powers of the natural numbers from 1 to $k-1$, that is, $s_n(k) = 1^n + 2^n + \dots + (k-1)^n$. Set up the equality

$$k^n = 1 + C_n^{n-1} s_{n-1}(k) + C_n^{n-2} s_{n-2}(k) + \dots + C_n^1 s_1(k) + s_0(k)$$

and show that

$$s_{n-1}(k) = \frac{1}{n!} \begin{vmatrix} k^n & C_n^{n-1} & C_n^{n-2} & \dots & C_n^1 & 1 \\ k^{n-1} & C_{n-1}^{n-2} & C_{n-1}^{n-3} & \dots & C_{n-1}^1 & 1 \\ k^{n-2} & 0 & C_{n-2}^{n-3} & \dots & C_{n-2}^1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ k^2 & 0 & 0 & \dots & C_2^1 & 1 \\ k & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

*605. Express as a determinant the nth coefficient l_n of the expansion $\frac{\tan x}{x} = 1 + l_1 x^2 + l_2 x^4 + \dots + l_n x^{2n} + \dots$.

606. Express as a determinant the nth coefficient f_n of the expansion $x \cot x = 1 - f_1 x^2 - f_2 x^4 - \dots - f_n x^{2n} - \dots$.

*607. Express as a determinant the nth coefficient a_n of the expansion $e^{-x} = 1 - a_1x + a_2x^2 - a_3x^3 + \ldots$, and then from it find the value of the determinant.

Sec. 10. The Rank of a Matrix. The Linear Dependence of Vectors and Linear Forms

Find the ranks of the following matrices by the method of bordered minors:

608. $\begin{pmatrix} 2 & -1 & 3 & -2 & 4 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{pmatrix}$.

609. $\begin{pmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -3 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{pmatrix}$.

610. $\begin{pmatrix} 3 & -1 & 3 & 2 & 5 \\ 5 & -3 & 2 & 3 & 4 \\ 1 & -3 & -5 & 0 & -7 \\ 7 & -5 & 1 & 4 & 1 \end{pmatrix}$.

611. $\begin{vmatrix} 4 & 3 & -5 & 2 & 3 \\ 8 & 6 & -7 & 4 & 2 \\ 4 & 3 & -8 & 2 & 7 \\ 4 & 3 & 1 & 2 & -5 \\ 8 & 6 & -1 & 4 & -6 \end{vmatrix}$.

612. Find the values of λ for which the matrix

$$\begin{pmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}$$

has the lowest rank.

What is the rank for the values of λ thus found, and what is it for other values of λ?

613. What is the rank of the matrix

$$\begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix}$$

for various values of λ?

614. Let A be a matrix of rank r and let M_k be a kth-order minor in the upper left-hand corner of A. Prove that by interchanging rows and interchanging columns it is pos-

sible to attain fulfillment of the conditions $M_1 \neq 0$, $M_2 \neq \neq 0$, . . ., $M_r \neq 0$, whereas all minors of order greater than r (if such exist) are equal to zero.

615. *Elementary transformations (or operations) of a matrix* are:

(1) multiplication of a row (or column) by a scalar different from zero;

(2) addition to a row (or a column) of another row (or column) multiplied by any scalar;

(3) interchanging any two rows (or columns).

Prove that elementary transformations do not alter the rank of a matrix.

616. Prove that an interchange of rows (or columns) of a matrix can be obtained by performing transformations of rows and columns solely of types (1) and (2) of problem 615.

617. Prove that by using elementary transformations (indicated in problem 615) it is possible to reduce any matrix of rank r to a form where the elements $a_{11} = a_{22} = \ldots$ $\ldots = a_{rr} = 1$ and all other elements are zero.

618. Prove that by applying elementary transformations to rows alone or columns alone it is possible to reduce a square matrix to triangular form where all elements on one side of the principal diagonal are zero; what is more, the zeros can be obtained either above or below the principal diagonal.

Using elementary transformations, compute the ranks of the following matrices:

619. $\begin{pmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{pmatrix}$.

620. $\begin{pmatrix} 47 & -67 & 35 & 201 & 155 \\ 26 & 98 & 23 & -294 & 86 \\ 16 & -428 & 1 & 1284 & 52 \end{pmatrix}$.

621. $\begin{pmatrix} 24 & 19 & 36 & 72 & -38 \\ 49 & 40 & 73 & 147 & -80 \\ 73 & 59 & 98 & 219 & -118 \\ 47 & 36 & 71 & 141 & -72 \end{pmatrix}$.

622. $\begin{pmatrix} 17 & -28 & 45 & 11 & 39 \\ 24 & -37 & 61 & 13 & 50 \\ 25 & -7 & 32 & -18 & -11 \\ 31 & 12 & 19 & -43 & -55 \\ 42 & 13 & 29 & -55 & -68 \end{pmatrix}$.

623. Prove that if a matrix contains m rows and is of rank r, then any s rows form a matrix whose rank is at least $r + s - m$.

624. Prove that adjoining a row (or a column) to a matrix either fails to alter the rank or increases it by unity.

625. Prove that crossing out one row (or one column) of a matrix fails to alter the rank if and only if the deleted row (or column) can be expressed linearly in terms of the remaining rows (or columns).

626. The *sum of two matrices* with the same number of rows and the same number of columns is a matrix each element of which is equal to the sum of the corresponding elements of the given matrices, that is, $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$. Prove that the rank of the sum of two matrices does not exceed the sum of their ranks.

627. Prove that any matrix of rank r can be expressed as a sum r of matrices of rank unity, but cannot be expressed as the sum of fewer than r such matrices.

628. Prove that if the rank of a matrix A remains unchanged after adjoining to it each column of matrix B with the same number of rows, then it will remain unchanged after adjoining to A all the columns of B.

***629.** Prove that if the rank of a matrix A is equal to r, then the minor d located at the intersection of any r linearly independent rows and r linearly independent columns of the matrix is nonzero.

***630.** Let A be a square matrix of order $n > 1$ and let $\hat{A}$ be the adjoint of A. Determine what changes will occur in the rank $\hat{r}$ of $\hat{A}$ when the rank r of matrix A is changed.

***631.** Prove that computing the rank of a symmetric matrix reduces to computing the principal minors alone, that is, the minors located in rows and columns with respectively identical labels. Namely, prove that

(1) if in a symmetric matrix A of order n there is a principal minor M_r of order r different from zero for which all bordering principal minors of orders $(r + 1)$ and $(r + 2)$

are equal to zero, then the rank of A is r (if all principal minors are zero, then we can assume the principal minor of zeroth order, M_0, to be equal to unity, and the theorem holds; for $r = n - 1$ there are no minors of order $r + 2$, but the theorem holds because the rank of A is equal to $n - 1$);

(2) the rank of a symmetric matrix is equal to the highest order of nonzero principal minors of that matrix.

*632. Let A be a symmetric matrix of rank r and let M_k be a minor of the kth order located in the upper left-hand corner of A. (For $k = 0$ we assume $M_0 = 1$.) Prove that by an appropriate interchange of rows and a corresponding interchange of columns of matrix A it is possible to attain a situation in which in the sequence of minors $M_0 = 1$, M_1, M_2, . . ., M_r no two adjacent ones are zero and $M_r \neq 0$, but minors of order greater than r (if such exist) are equal to zero.

*633. Prove that the rank of a skew-symmetric matrix is determined by its principal minors, namely:

(1) if there is a nonzero principal minor of order r for which all bordering principal minors of order $r + 2$ are zero, then the rank of the matrix is equal to r;

(2) the rank of a skew-symmetric matrix is equal to the highest order of the nonzero principal minors of the matrix.

*634. Let A be a skew-symmetric matrix of rank r and let M_k be the kth-order minor located in the upper left-hand corner of matrix A ($M_0 = 1$). Prove that an appropriate interchange in rows and a corresponding interchange in columns of A results in the minors M_0, M_2, M_4, . . ., M_r being nonzero and the minors M_1, M_3, . . ., M_{r-1} and all minors of order above r (if such exist) being zero.

635. Prove that the rank of a skew-symmetric matrix is an even number.

636. Find the linear combination $3a_1 + 5a_2 - a_3$ of vectors

$$a_1 = (4, 1, 3, -2), \quad a_2 = (1, 2, -3, 2),$$

$$a_3 = (16, 9, 1, -3).$$

637. Find the vector x from the equation

$$a_1 + 2a_2 + 3a_3 + 4x = 0,$$

where

$$a_1 = (5, -8, -1, 2), \quad a_2 = (2, -4, 4, -3),$$
$$a_3 = (-3, 2, -5, 4).$$

638. Find the vector x from the equation

$$3(a_1 - x) + 2(a_2 + x) = 5(a_3 + x),$$

where

$$a_1 = (2, 5, 1, 3), \quad a_2 = (10, 1, 5, 10),$$
$$a_3 = (4, 1, -1, 1).$$

Determine whether the following sets of vectors are linearly dependent or linearly independent:

639. $a_1 = (1, 2, 3)$, $a_2 = (3, 6, 7)$.

640. $a_1 = (4, -2, 6)$, $a_2 = (6, -3, 9)$.

641. $a_1 = (2, -3, 1)$, $a_2 = (3, -1, 5)$, $a_3 = (1, -4, 3)$.

642. $a_1 = (5, 4, 3)$, $a_2 = (3, 3, 2)$, $a_3 = (8, 1, 3)$.

643. $a_1 = (4, -5, 2, 6)$, $a_2 = (2, -2, 1, 3)$, $a_3 = (6, -3, 3, 9)$, $a_4 = (4, -1, 5, 6)$.

644. $a_1 = (1, 0, 0, 2, 5)$, $a_2 = (0, 1, 0, 3, 4)$, $a_3 = (0, 0, 1, 4, 7)$, $a_4 = (2, -3, 4, 11, 12)$.

645. If from among the coordinates (components) of each vector of a given set of vectors of the same number of dimensions we choose coordinates located at definite sites (which are the same for all vectors) and if we retain the order, then we obtain a second set of vectors, which will be said to be *shortened* relative to the first set, which will be said to be *extended* relative to the second set. Prove that any shortened set for a linearly dependent set of vectors is itself linearly dependent, and any extended set for a linearly independent set of vectors is itself linearly independent.

646. Prove that a set of vectors having two equal vectors is linearly dependent.

647. Prove that a set of vectors, two of which differ by a scalar factor, is linearly dependent.

648. Prove that a set of vectors containing the zero vector is linearly dependent.

649. Prove that if a portion of a set of vectors is linearly dependent, then the whole set is linearly dependent.

650. Prove that any part of a linearly independent set of vectors is itself linearly independent.

*651. Given a set of vectors

$$a_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}) \qquad (i = 1, 2, \ldots, s;\ s \leqslant n).$$

Prove that if $|\alpha_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{s} |\alpha_{ij}|$, then the given set of vectors is linearly independent.

652. Prove that if three vectors a_1, a_2, a_3 are linearly dependent and the vector a_3 is not expressible linearly in terms of the vectors a_1 and a_2, then the vectors a_1 and a_2 differ only by a numerical factor.

653. Prove that if the vectors $a_1, a_2, \ldots, a_k$ are linearly independent and the vectors $a_1, a_2, \ldots, a_k, b$ are linearly dependent, then b is linearly expressible in terms of the vectors $a_1, a_2, \ldots, a_k$.

654. Use the preceding problem to prove that every vector of a given set of vectors is linearly expressible in terms of any linearly independent subset (of the given set) that contains the maximum number of vectors.

655. Prove that an ordered set of nonzero vectors $a_1, a_2, \ldots, a_k$ is linearly independent if and only if not a single vector can be expressed linearly in terms of the preceding ones.

*656. Prove that if in front of an ordered linearly independent set of vectors $a_1, a_2, \ldots, a_k$ we adjoin some vector b, then not more than one vector of the resulting set will be expressible linearly in terms of the preceding ones.

*657. Prove that if the vectors $a_1, a_2, \ldots, a_r$ are linearly independent and can be expressed linearly in terms of the vectors $b_1, b_2, \ldots, b_s$, then $r \leqslant s$.

*658. The *basis of a given set of vectors* is a subset with the following properties:

(1) the subset is linearly independent;

(2) any vector of the whole set can be linearly expressed in terms of the vectors of the subset.

Prove that

(a) all bases of a given set of vectors contain the same number of vectors;

(b) the number of vectors of any basis is the maximum number of linearly independent vectors of the given set; this number is termed the rank of the original set;

(c) if a given set of vectors has rank r, then any r linearly independent vectors form a basis of that set of vectors.

*659. Prove that any linearly independent subset of a given set of vectors can be completed to form a basis of the set.

660. Two sets of vectors are said to be *equivalent* if each vector of one set can be linearly expressed in terms of the vectors of the other and vice versa. Prove that two equivalent linearly independent sets of vectors contain the same number of vectors.

661. Prove that if the vectors $a_1, a_2, \ldots, a_k$ are linearly expressed in terms of the vectors $b_1, b_2, \ldots, b_l$, then the rank of the first set does not exceed the rank of the second set.

662. Given the vectors:

$$a_1 = (0, 1, 0, 2, 0), \quad a_2 = (7, 4, 1, 8, 3),$$
$$a_3 = (0, 3, 0, 4, 0), \quad a_4 = (1, 9, 5, 7, 1),$$
$$a_5 = (0, 1, 0, 5, 0).$$

Is it possible to choose the numbers c_{ij} ($i, j = 1, 2, \ldots, 5$) so that the vectors

$$b_1 = c_{11}a_1 + c_{12}a_2 + c_{13}a_3 + c_{14}a_4 + c_{15}a_5,$$
$$b_2 = c_{21}a_1 + c_{22}a_2 + c_{23}a_3 + c_{24}a_4 + c_{25}a_5,$$
$$b_3 = c_{31}a_1 + c_{32}a_2 + c_{33}a_3 + c_{34}a_4 + c_{35}a_5,$$
$$b_4 = c_{41}a_1 + c_{42}a_2 + c_{43}a_3 + c_{44}a_4 + c_{45}a_5,$$
$$b_5 = c_{51}a_1 + c_{52}a_2 + c_{53}a_3 + c_{54}a_4 + c_{55}a_5$$

are linearly independent?

663. Prove that the vector b can be linearly expressed in terms of the vectors $a_1, a_2, \ldots, a_k$ if and only if the rank of this set of vectors remains unchanged when the vector b is adjoined to the set.

*664. Prove that

(1) two equivalent sets of vectors have the same rank;

(2) the converse of (1) is erroneous.

However, the following assertion holds true:

(3) if two sets of vectors have the same rank and one of the sets is linearly expressible in terms of the other, then the sets are equivalent.

Find all values of λ for which the vector b can be linearly expressed in terms of the vectors $a_1, a_2, \ldots, a_s$:

665. $a_1 = (2, 3, 5)$,
$a_2 = (3, 7, 8)$,
$a_3 = (1, -6, 1)$,
$b = (7, -2, \lambda)$.

666. $a_1 = (4, 4, 3)$,
$a_2 = (7, 2, 1)$,
$a_3 = (4, 1, 6)$,
$b = (5, 9, \lambda)$.

667. $a_1 = (3, 4, 2)$,
$a_2 = (6, 8, 7)$,
$b = (9, 12, \lambda)$.

668. $a_1 = (3, 2, 5)$,
$a_2 = (2, 4, 7)$,
$a_3 = (5, 6, \lambda)$,
$b = (1, 3, 5)$.

669. $a_1 = (3, 2, 6)$,
$a_2 = (7, 3, 9)$,
$a_3 = (5, 1, 3)$,
$b = (\lambda, 2, 5)$.

670. Explain the answers to problems 665-669 from the standpoint of the configuration of the given vectors in space.

671. Use problem 657 to prove that more than n n-dimensional vectors are always linearly dependent.

672. Find all maximum linearly independent subsets of the following sets of vectors:

$a_1 = (4, -1, 3, -2)$, $a_2 = (8, -2, 6, -4)$,
$a_3 = (3, -1, 4, -2)$, $a_4 = (6, -2, 8, -4)$.

Find all bases of the following sets of vectors:

673. $a_1 = (1, 2, 0, 0)$,
$a_2 = (1, 2, 3, 4)$,
$a_3 = (3, 6, 0, 0)$.

674. $a_1 = (1, 2, 3, 4)$,
$a_2 = (2, 3, 4, 5)$,
$a_3 = (3, 4, 5, 6)$,
$a_4 = (4, 5, 6, 7)$.

675. $a_1 = (2, 1, -3, 1)$,
$a_2 = (4, 2, -6, 2)$,
$a_3 = (6, 3, -9, 3)$,
$a_4 = (1, 1, 1, 1)$.

676. $a_1 = (1, 2, 3)$,
$a_2 = (2, 3, 4)$,
$a_3 = (3, 2, 3)$,
$a_4 = (4, 3, 4)$,
$a_5 = (1, 1, 1)$.

677. In what case does a set of vectors have a unique basis?

678. How many bases does a set of $k + 1$ vectors of rank k have that contains proportional nonzero vectors?

Find some basis of the given set of vectors and express in terms of the vectors of the basis all the vectors of the set that do not appear in the given basis:

679. $a_1 = (5, 2, -3, 1)$,
$a_2 = (4, 1, -2, 3)$,
$a_3 = (1, 1, -1, -2)$,
$a_4 = (3, 4, -1, 2)$.

680. $a_1 = (2, -1, 3, 5)$,
$a_2 = (4, -3, 1, 3)$,
$a_3 = (3, -2, 3, 4)$,
$a_4 = (4, -1, 15, 17)$,
$a_5 = (7, -6, -7, 0)$.

681. $a_1 = (1, 2, 3, -4)$,
$a_2 = (2, 3, -4, 1)$,
$a_3 = (2, -5, 8, -3)$,
$a_4 = (5, 26, -9, -12)$,
$a_5 = (3, -4, 1, 2)$.

*682. Given a set of vectors $x_1, x_2, \ldots, x_n$, all having the same number of dimensions. The basic system of linear relations of this set of vectors is a system of relations of the form

$$\sum_{j=1}^{n} \alpha_{i,j} x_j = 0 \qquad (i = 1, 2, \ldots, s),$$

which has the following two properties:

(a) the system of relations is linearly independent, which means that the set of vectors

$$a_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}) \qquad (i = 1, 2, \ldots, s)$$

is linearly independent;

(b) any linear dependence of the vectors $x_1, x_2, \ldots, x_n$ is a consequence of the relations of the system, that is, if $\sum_{j=1}^{n} \alpha_j x_j = 0$, then the vector $a = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a linear combination of the vectors $a_1, a_2, \ldots, a_s$. Prove that

(1) if $x_1, x_2, \ldots, x_r$ is a basis of the given set of vectors and $x_i = \sum_{j=1}^{r} \lambda_{i,j} x_j$, $i = r+1, r+2, \ldots, n$, then one of the basic systems of linear relations of the given set of vectors is the system of relations $x_i - \sum_{j=1}^{r} \lambda_{i,j} x_j = 0$ $(i = r+1, r+2, \ldots, n)$;

(2) all basic systems of linear relations contain the same number of relations;

(3) if some basic system of linear relations contains s relations, then any system of s linearly independent linear relations of the same set of vectors is also a basic system of linear relations;

(4) if the system of relations $\sum_{j=1}^{n} \alpha_{i,j} x_j = 0 \ (i = 1, 2, \ldots, s)$ is a basic system of linear relations, then the system of relations $\sum_{j=1}^{n} \beta_{i,j} x_j = 0 \quad (i = 1, 2, \ldots, s)$ will be a basic system of linear relations if and only if, assuming $a_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}), \quad i = 1, 2, \ldots, s, \quad b_i = (\beta_{i,1} \beta_{i,2}, \ldots, \beta_{i,n}), \quad i = 1, 2, \ldots, s$, we have

$$b_i = \sum_{j=1}^{s} \gamma_{i,j} a_j \quad (i = 1, 2, \ldots, s),$$

where the coefficients $\gamma_{i,j}$ form a nonzero determinant of order s.

Using multiplication of matrices, we can write the last s vector equations as a single matrix equation:

$B = CA$, where $A = (\alpha_{i,j})_{s,n}$, $B = (\beta_{i,j})_{s,n}$ and $C = (\gamma_{i,j})_s$.

C is a nonsingular matrix of order s.

Define the basic system of linear relations for a set of linear forms in the manner of problem 682 for a set of vectors, and then find a basic system of linear relations for the following systems of linear forms:

683. $f_1 = 5x_1 - 3x_2 + 2x_3 + 4x_4,$
$f_2 = 2x_1 - x_2 + 3x_3 + 5x_4,$
$f_3 = 4x_1 - 3x_2 - 5x_3 - 7x_4,$
$f_4 = x_1 + 7x_2 + 11x_4.$

684. $f_1 = 8x_1 + 7x_2 + 4x_3 + 5x_4,$
$f_2 = 3x_1 + 2x_2 + x_3 + 4x_4,$
$f_3 = 2x_1 + 3x_2 + 2x_3 - 3x_4,$
$f_4 = x_1 - x_2 - x_3 + 7x_4,$
$f_5 = 5x_2 + 4x_3 - 17x_4.$

685. $f_1 = 5x_1 + 2x_2 - x_3 + 3x_4 + 4x_5,$
$f_2 = 3x_1 + x_2 - 2x_3 + 3x_4 + 5x_5,$

$$f_3 = 6x_1 + 3x_2 - 2x_3 + 4x_4 + 7x_5,$$
$$f_4 = 7x_1 + 4x_2 - 3x_3 + 2x_4 + 4x_5.$$

686. $f_1 = 2x_1 - 3x_2 + 4x_3 - 5x_4,$
$f_2 = x_1 - 2x_2 + 7x_3 - 8x_4,$
$f_3 = 3x_1 - 4x_2 + x_3 - 2x_4,$
$f_4 = 4x_1 - 5x_2 + 6x_3 - 7x_4,$
$f_5 = 6x_1 - 7x_2 - x_4.$

687. $f_1 = 3x_1 + 2x_2 - 2x_3 - x_4 + 4x_5,$
$f_2 = 6x_1 + 4x_2 - 4x_3 - 2x_4 + 8x_5,$
$f_3 = 7x_1 + 5x_2 - 3x_3 - 2x_4 + x_5,$
$f_4 = 4x_1 + 4x_2 - 4x_3 - 3x_4 + 5x_5.$
$f_5 = 8x_1 + 7x_2 - 5x_3 - 4x_4 + 2x_5.$

*688. Given a system of linear forms:

$$f_j = \sum_{k=1}^{n} a_{j,k}x_k \qquad (j = 1, 2, \ldots, s) \tag{1}$$

and a second system of linear forms that are linearly dependent on the forms of the first system:

$$\varphi_i = \sum_{j=1}^{s} c_{ij}f_j \qquad (i = 1, 2, \ldots, t). \tag{2}$$

Prove that the rank of the system of forms (2) does not exceed the rank of (1). If $s = t$ and the determinant $|c_{ij}|_s$ is nonzero, then the ranks of both systems of linear forms coincide.

Sec. 11. Systems of Linear Equations

Investigate the following systems of equations for consistency and find the general solution and one particular solution in each system:

689. $2x_1 + 7x_2 + 3x_3 + x_4 = 6,$
$3x_1 + 5x_2 + 2x_3 + 2x_4 = 4,$
$9x_1 + 4x_2 + x_3 + 7x_4 = 2.$

690. $2x_1 - 3x_2 + 5x_3 + 7x_4 = 1,$
$4x_1 - 6x_2 + 2x_3 + 3x_4 = 2,$
$2x_1 - 3x_2 - 11x_3 - 15x_4 = 1.$

691. $3x_1 + 4x_2 + x_3 + 2x_4 = 3,$
$6x_1 + 8x_2 + 2x_3 + 5x_4 = 7,$
$9x_1 + 12x_2 + 3x_3 + 10x_4 = 13.$

692. $3x_1 - 5x_2 + 2x_3 + 4x_4 = 2,$
$7x_1 - 4x_2 + x_3 + 3x_4 = 5,$
$5x_1 + 7x_2 - 4x_3 - 6x_4 = 3.$

693. $2x_1 + 5x_2 - 8x_3 = 8,$
$4x_1 + 3x_2 - 9x_3 = 9,$
$2x_1 + 3x_2 - 5x_3 = 7,$
$x_1 + 8x_2 - 7x_3 = 12.$

694. $3x_1 - 2x_2 + 5x_3 + 4x_4 = 2,$
$6x_1 - 4x_2 + 4x_3 + 3x_4 = 3,$
$9x_1 - 6x_2 + 3x_3 + 2x_4 = 4.$

695. $2x_1 - x_2 + 3x_3 - 7x_4 = 5,$
$6x_1 - 3x_2 + x_3 - 4x_4 = 7,$
$4x_1 - 2x_2 + 14x_3 - 31x_4 = 18.$

696. $9x_1 - 3x_2 + 5x_3 + 6x_4 = 4,$
$6x_1 - 2x_2 + 3x_3 + x_4 = 5,$
$3x_1 - x_2 + 3x_3 + 14x_4 = -8.$

697. $3x_1 + 2x_2 + 2x_3 + 2x_4 = 2,$
$2x_1 + 3x_2 + 2x_3 + 5x_4 = 3,$
$9x_1 + x_2 + 4x_3 - 5x_4 = 1,$
$2x_1 + 2x_2 + 3x_3 + 4x_4 = 5,$
$7x_1 + x_2 + 6x_3 - x_4 = 7.$

698. $x_1 + x_2 + 3x_3 - 2x_4 + 3x_5 = 1,$
$2x_1 + 2x_2 + 4x_3 - x_4 + 3x_5 = 2,$
$3x_1 + 3x_2 + 5x_3 - 2x_4 + 3x_5 = 1,$
$2x_1 + 2x_2 + 8x_3 - 3x_4 + 9x_5 = 2.$

699. $2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2,$
$6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3,$
$6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 = 9,$
$4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1.$

700.
$$\begin{aligned} 6x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_5 &= 1,\\ 3x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 &= 3,\\ 3x_1 + 2x_2 - 2x_3 + x_4 &= -7,\\ 9x_1 + 6x_2 + x_3 + 3x_4 + 2x_5 &= 2. \end{aligned}$$

701.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 &= 4,\\ 3x_1 + 6x_2 + 5x_3 - 4x_4 + 3x_5 &= 5,\\ x_1 + 2x_2 + 7x_3 - 4x_4 + x_5 &= 11,\\ 2x_1 + 4x_2 + 2x_3 - 3x_4 + 3x_5 &= 6. \end{aligned}$$

702.
$$\begin{aligned} 6x_1 + 3x_2 + 2x_3 + 3x_4 + 4x_5 &= 5,\\ 4x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= 4,\\ 4x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 &= 0,\\ 2x_1 + x_2 + 7x_3 + 3x_4 + 2x_5 &= 1. \end{aligned}$$

703.
$$\begin{aligned} 8x_1 + 6x_2 + 5x_3 + 2x_4 &= 21,\\ 3x_1 + 3x_2 + 2x_3 + x_4 &= 10,\\ 4x_1 + 2x_2 + 3x_3 + x_4 &= 8,\\ 3x_1 + 5x_2 + x_3 + x_4 &= 15,\\ 7x_1 + 4x_2 + 5x_3 + 2x_4 &= 18. \end{aligned}$$

704.
$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 2x_4 &= 4,\\ 4x_1 + 3x_2 + x_3 + x_4 &= 5,\\ 5x_1 + 11x_2 + 3x_3 + 2x_4 &= 2,\\ 2x_1 + 5x_2 + x_3 + x_4 &= 1,\\ x_1 - 7x_2 - x_3 + 2x_4 &= 7. \end{aligned}$$

***705.** Prove that

(a) any system of s linear equations in n unknowns, whose matrix of the coefficients of the unknowns has rank r, can, via a change in the numbering of the equations and unknowns, be brought to the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad (i = 1, 2, \ldots, s), \tag{1}$$

which has the properties that

$$m_0 = 1, \quad m_1 \neq 0, \quad m_2 \neq 0, \ \ldots, \ m_r \neq 0, \tag{2}$$

where m_k is a kth-order minor in the upper left-hand corner of the matrix of coefficients of the unknowns of the system (1);

(b) the system of equations (1) having the properties (2) can—via a number of subtractions of its equations, which are multiplied by suitable numbers, from subsequent equations—be reduced to the equivalent system

$$\sum_{j=1}^{n} c_{ij}x_j = d_i \qquad (i = 1, 2, \ldots, s), \tag{3}$$

which has the following properties:

$$\left.\begin{array}{l} c_{ii} \neq 0 \quad \text{for } i = 1, 2, \ldots, r \\ c_{ij} = 0 \quad \text{for } j < i \leqslant r \text{ and also} \\ \qquad\quad \text{for } i > r \text{ and } j = 1, 2, \ldots, n. \end{array}\right\} \tag{4}$$

If $d_i = 0$ for $i = r + 1, r + 2, \ldots, s$, then the systems (3) and (1) are consistent, and when $r = n$ there is a unique solution; when $r < n$ there is an infinity of solutions.

In the latter instance, $x_{r+1}, \ldots, x_n$ are free unknowns. Using the rth equation, we can express x_r in terms of free unknowns. If this expression is put into the $(r - 1)$th equation, we find an expression of x_{r-1} in terms of the free unknowns, and so on.

Finally, using the first equation we find an expression for x_1 in terms of the free unknowns.

The resulting expressions of $x_1, x_2, \ldots, x_r$ in terms of the free unknowns $x_{r+1}, \ldots, x_n$ constitute the general solution of the systems (3) and (1). This means that for arbitrary values of the free unknowns we obtain from the expressions thus found the solutions of systems (3) and (1), and any solution of these systems can be obtained in this manner for suitable values of the free unknowns.

If $d_i \neq 0$ for even one value of $i > r$, then the systems (3) and (1) are not consistent.

The foregoing method of investigating and solving systems of linear equations is known as the elimination method (compare with problem 565).

Using the method of elimination given in problem 705, test for consistency and find the general solution of the following systems of equations (if the original system has integral coefficients, then fractions will be avoided in the elimination process):

706. $x_1 + 2x_2 + 3x_3 + x_4 = 3,$
$x_1 + 4x_2 + 5x_3 + 2x_4 = 2,$

$$2x_1 + 9x_2 + 8x_3 + 3x_4 = 7,$$
$$3x_1 + 7x_2 + 7x_3 + 2x_4 = 12,$$
$$5x_1 + 7x_2 + 9x_3 + 2x_4 = 20.$$

707. $$12x_1 + 14x_2 - 15x_3 + 23x_4 + 27x_5 = 5,$$
$$16x_1 + 18x_2 - 22x_3 + 29x_4 + 37x_5 = 8,$$
$$18x_1 + 20x_2 - 21x_3 + 32x_4 + 41x_5 = 9,$$
$$10x_1 + 12x_2 - 16x_3 + 20x_4 + 23x_5 = 4.$$

708. $$10x_1 + 23x_2 + 17x_3 + 44x_4 = 25,$$
$$15x_1 + 35x_2 + 26x_3 + 69x_4 = 40,$$
$$25x_1 + 57x_2 + 42x_3 + 108x_4 = 65,$$
$$30x_1 + 69x_2 + 51x_3 + 133x_4 = 95.$$

709. $$45x_1 - 28x_2 + 34x_3 - 52x_4 = 9,$$
$$36x_1 - 23x_2 + 29x_3 - 43x_4 = 3,$$
$$35x_1 - 21x_2 + 28x_3 - 45x_4 = 16,$$
$$47x_1 - 32x_2 + 36x_3 - 48x_4 = -17,$$
$$27x_1 - 19x_2 + 22x_3 - 35x_4 = 6.$$

710. $$12x_2 - 16x_3 + 25x_4 = 29,$$
$$27x_1 + 24x_2 - 32x_3 + 47x_4 = 55,$$
$$50x_1 + 51x_2 - 68x_3 + 95x_4 = 115,$$
$$31x_1 + 21x_2 - 28x_3 + 46x_4 = 50.$$

711. $$24x_1 + 14x_2 + 30x_3 + 40x_4 + 41x_5 = 28,$$
$$36x_1 + 21x_2 + 45x_3 + 61x_4 + 62x_5 = 43,$$
$$48x_1 + 28x_2 + 60x_3 + 82x_4 + 83x_5 = 58,$$
$$60x_1 + 35x_2 + 75x_3 + 99x_4 + 102x_5 = 69.$$

Investigate the following systems of equations and find the general solutions, depending on the value of the parameter λ:

712. $$5x_1 - 3x_2 + 2x_3 + 4x_4 = 3,$$
$$4x_1 - 2x_2 + 3x_3 + 7x_4 = 1,$$
$$8x_1 - 6x_2 - x_3 - 5x_4 = 9,$$
$$7x_1 - 3x_2 + 7x_3 + 17x_4 = \lambda.$$

713. $3x_1 + 2x_2 + 5x_3 + 4x_4 = 3,$
$2x_1 + 3x_2 + 6x_3 + 8x_4 = 5,$
$x_1 - 6x_2 - 9x_3 - 20x_4 = -11,$
$4x_1 + x_2 + 4x_3 + \lambda x_4 = 2.$

714. $2x_1 + 5x_2 + x_3 + 3x_4 = 2,$
$4x_1 + 6x_2 + 3x_3 + 5x_4 = 4,$
$4x_1 + 14x_2 + x_3 + 7x_4 = 4,$
$2x_1 - 3x_2 + 3x_3 + \lambda x_4 = 7.$

715. $2x_1 - x_2 + 3x_3 + 4x_4 = 5,$
$4x_1 - 2x_2 + 5x_3 + 6x_4 = 7,$
$6x_1 - 3x_2 + 7x_3 + 8x_4 = 9,$
$\lambda x_1 - 4x_2 + 9x_3 + 10x_4 = 11.$

716. $2x_1 + 3x_2 + x_3 + 2x_4 = 3,$
$4x_1 + 6x_2 + 3x_3 + 4x_4 = 5,$
$6x_1 + 9x_2 + 5x_3 + 6x_4 = 7,$
$8x_1 + 12x_2 + 7x_3 + \lambda x_4 = 9.$

717. $\lambda x_1 + x_2 + x_3 = 1,$
$x_1 + \lambda x_2 + x_3 = 1,$
$x_1 + x_2 + \lambda x_3 = 1.$

718. $\lambda x_1 + x_2 + x_3 + x_4 = 1,$
$x_1 + \lambda x_2 + x_3 + x_4 = 1,$
$x_1 + x_2 + \lambda x_3 + x_4 = 1,$
$x_1 + x_2 + x_3 + \lambda x_4 = 1.$

719. $(1 + \lambda) x_1 + x_2 + x_3 = 1,$
$x_1 + (1 + \lambda) x_2 + x_3 = \lambda,$
$x_1 + x_2 + (1 + \lambda) x_3 = \lambda^2.$

720. $(\lambda + 1) x_1 + x_2 + x_3 = \lambda^2 + 3\lambda,$
$x_1 + (\lambda + 1) x_2 + x_3 = \lambda^3 + 3\lambda^2,$
$x_1 + x_2 + (\lambda + 1) x_3 = \lambda^4 + 3\lambda^3.$

Investigate the following systems of equations and find the general solutions, depending on the values that enter into the coefficients of the parameters:

721. $\begin{aligned} x + y + z &= 1, \\ ax + by + cz &= d, \\ a^2x + b^2y + c^2z &= d^2. \end{aligned}$

722. $\begin{aligned} ax + y + z &= 1, \\ x + by + z &= 1, \\ x + y + cz &= 1. \end{aligned}$

In what case are zero values of some of the unknowns possible?

*723. $\begin{aligned} ax + y + z &= a, \\ x + by + z &= b, \\ x + y + cz &= c. \end{aligned}$

Find the general solution and the fundamental system (or set) of solutions for the following systems of equations:

724. $\begin{aligned} x_1 + 2x_2 + 4x_3 - 3x_4 &= 0, \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0, \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 &= 0, \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0. \end{aligned}$

725. $\begin{aligned} 2x_1 - 4x_2 + 5x_3 + 3x_4 &= 0, \\ 3x_1 - 6x_2 + 4x_3 + 2x_4 &= 0, \\ 4x_1 - 8x_2 + 17x_3 + 11x_4 &= 0. \end{aligned}$

726. $\begin{aligned} 3x_1 + 2x_2 + x_3 + 3x_4 + 5x_5 &= 0, \\ 6x_1 + 4x_2 + 3x_3 + 5x_4 + 7x_5 &= 0, \\ 9x_1 + 6x_2 + 5x_3 + 7x_4 + 9x_5 &= 0, \\ 3x_1 + 2x_2 + 4x_4 + 8x_5 &= 0. \end{aligned}$

727. $\begin{aligned} 3x_1 + 5x_2 + 2x_3 &= 0, \\ 4x_1 + 7x_2 + 5x_3 &= 0, \\ x_1 + x_2 - 4x_3 &= 0, \\ 2x_1 + 9x_2 + 6x_3 &= 0. \end{aligned}$

728. $\begin{aligned} 6x_1 - 2x_2 + 2x_3 + 5x_4 + 7x_5 &= 0, \\ 9x_1 - 3x_2 + 4x_3 + 8x_4 + 9x_5 &= 0, \\ 6x_1 - 2x_2 + 6x_3 + 7x_4 + x_5 &= 0, \\ 3x_1 - x_2 + 4x_3 + 4x_4 - x_5 &= 0. \end{aligned}$

729. $\begin{aligned} x_1 - x_3 &= 0, \\ x_2 - x_4 &= 0, \\ -x_1 + x_3 - x_5 &= 0, \end{aligned}$

730. $\begin{aligned} x_1 - x_3 + x_5 &= 0, \\ x_2 - x_4 + x_6 &= 0, \\ x_1 - x_2 + x_5 - x_6 &= 0. \end{aligned}$

$$-x_2 + x_4 - x_6 = 0, \qquad x_2 - x_3 + x_6 = 0,$$
$$-x_3 + x_5 = 0, \qquad x_1 - x_4 + x_5 = 0.$$
$$-x_4 + x_6 = 0.$$

731.
$$\begin{aligned} 5x_1 + 6x_2 - 2x_3 + 7x_4 + 4x_5 &= 0,\\ 2x_1 + 3x_2 - x_3 + 4x_4 + 2x_5 &= 0,\\ 7x_1 + 9x_2 - 3x_3 + 5x_4 + 6x_5 &= 0,\\ 5x_1 + 9x_2 - 3x_3 + x_4 + 6x_5 &= 0. \end{aligned}$$

732.
$$\begin{aligned} 3x_1 + 4x_2 + x_3 + 2x_4 + 3x_5 &= 0,\\ 5x_1 + 7x_2 + x_3 + 3x_4 + 4x_5 &= 0,\\ 4x_1 + 5x_2 + 2x_3 + x_4 + 5x_5 &= 0,\\ 7x_1 + 10x_2 + x_3 + 6x_4 + 5x_5 &= 0. \end{aligned}$$

733. Prove that for any homogeneous system of linear equations with rational (and, as a particular case, integral) coefficients it is possible to construct an integral fundamental system of solutions (provided that the rank of the matrix of coefficients is less than the number of unknowns).

734. Prove that for the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \quad (i = 1, 2, \ldots, s) \tag{1}$$

of rank $r < n$, any $n - r$ linearly independent solutions

$$\begin{gathered} \alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n},\\ \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n},\\ \cdots\cdots\cdots\cdots\\ \alpha_{n-r,1}, \alpha_{n-r,2}, \ldots, \alpha_{n-r,n} \end{gathered}$$

form a fundamental system of solutions, and the general solution can be expressed in the form

$$x_j = \sum_{k=1}^{n-r} c_k \alpha_{kj} \qquad (j = 1, 2, \ldots, n), \tag{2}$$

where $c_1, c_2, \ldots, c_{n-r}$ are arbitrary parameters. In other words, prove that for any values of the parameters $c_1, c_2, \ldots, c_{n-r}$ formulas (2) yield a solution of system (1), and any solution of system (1) can be obtained from (2) for suitable values of the parameters $c_1, c_2, \ldots, c_{n-r}$.

For the following systems of equations find the general solution of form (2) of the preceding problem where each

unknown is given by a homogeneous linear expression of the parameters with integral coefficients:

735.
$$\begin{aligned} 2x_1 + x_2 - 4x_3 &= 0,\\ 3x_1 + 5x_2 - 7x_3 &= 0,\\ 4x_1 - 5x_2 - 6x_3 &= 0. \end{aligned}$$

736.
$$\begin{aligned} 2x_1 - x_2 + 5x_3 + 7x_4 &= 0,\\ 4x_1 - 2x_2 + 7x_3 + 5x_4 &= 0,\\ 2x_1 - x_2 + x_3 - 5x_4 &= 0. \end{aligned}$$

737.
$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 + 2x_4 + 7x_5 &= 0,\\ 6x_1 + 4x_2 + 7x_3 + 4x_4 + 5x_5 &= 0,\\ 3x_1 + 2x_2 - x_3 + 2x_4 - 11x_5 &= 0,\\ 6x_1 + 4x_2 + x_3 + 4x_4 - 13x_5 &= 0. \end{aligned}$$

738.
$$\begin{aligned} 6x_1 - 2x_2 + 3x_3 + 4x_4 + 9x_5 &= 0,\\ 3x_1 - x_2 + 2x_3 + 6x_4 + 3x_5 &= 0,\\ 6x_1 - 2x_2 + 5x_3 + 20x_4 + 3x_5 &= 0,\\ 9x_1 - 3x_2 + 4x_3 + 2x_4 + 15x_5 &= 0. \end{aligned}$$

739.
$$\begin{aligned} 2x_1 + 7x_2 + 4x_3 + 5x_4 + 8x_5 &= 0,\\ 4x_1 + 4x_2 + 8x_3 + 5x_4 + 4x_5 &= 0,\\ x_1 - 9x_2 - 3x_3 - 5x_4 - 14x_5 &= 0,\\ 3x_1 + 5x_2 + 7x_3 + 5x_4 + 6x_5 &= 0. \end{aligned}$$

740.
$$\begin{aligned} 3x_1 + 4x_2 + 3x_3 + 9x_4 + 6x_5 &= 0,\\ 9x_1 + 8x_2 + 5x_3 + 6x_4 + 9x_5 &= 0,\\ 3x_1 + 8x_2 + 7x_3 + 30x_4 + 15x_5 &= 0,\\ 6x_1 + 6x_2 + 4x_3 + 7x_4 + 5x_5 &= 0. \end{aligned}$$

741. Do the rows of each of the matrices

$$A = \begin{pmatrix} 30 & -24 & 43 & -50 & -5 \\ 9 & -15 & 8 & 5 & 2 \\ 4 & 2 & 9 & -20 & -3 \end{pmatrix},$$

$$B = \begin{pmatrix} 4 & 2 & 9 & -20 & -3 \\ 1 & -11 & 2 & 13 & 4 \\ 9 & -15 & 8 & 5 & 2 \end{pmatrix}$$

form a fundamental set of solutions for the system of equations

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 + x_4 + 6x_5 &= 0, \\ 5x_1 + 9x_2 + 7x_3 + 4x_4 + 7x_5 &= 0, \\ 4x_1 + 3x_2 - x_3 - x_4 + 11x_5 &= 0, \\ x_1 + 6x_2 + 8x_3 + 5x_4 - 4x_5 &= 0? \end{aligned}$$

742. Which of the rows of the matrix

$$\begin{pmatrix} 6 & 2 & 3 & -2 & -7 \\ 5 & 3 & 7 & -6 & -4 \\ 8 & 0 & -5 & 6 & -13 \\ 4 & -2 & -7 & 5 & -7 \end{pmatrix}$$

form a fundamental set of solutions for the system of equations

$$\begin{aligned} 2x_1 - 5x_2 + 3x_3 + 2x_4 + x_5 &= 0, \\ 5x_1 - 8x_2 + 5x_3 + 4x_4 + 3x_5 &= 0, \\ x_1 - 7x_2 + 4x_3 + 2x_4 &= 0, \\ 4x_1 - x_2 + x_3 + 2x_4 + 3x_5 &= 0? \end{aligned}$$

***743.** Prove that if in place of the free unknowns we put into the general solution of a homogeneous system of linear equations of rank r having n unknowns, where $r < n$, certain numbers seriately taken from each row of a nonzero determinant of order $n - r$ and if we find the appropriate values of the remaining unknowns, the result is a fundamental set of solutions, and conversely, any fundamental solution set of the given system of equations can be obtained in this manner for a suitable choice of a nonzero determinant of order $n - r$.

744. Suppose the rows of the matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{p1} & \alpha_{p2} & \dots & \alpha_{pn} \end{pmatrix}$$

form a fundamental solution set of a homogeneous system of linear equations of rank r with n unknowns ($n = r + p$).

Prove that the rows of the matrix

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{p1} & \beta_{p2} & \cdots & \beta_{pn} \end{pmatrix}$$

will also form a fundamental set of solutions of the same system of equations if and only if there exists a nonsingular matrix of order p

$$C = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1p} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{p1} & \gamma_{p2} & \cdots & \gamma_{pp} \end{pmatrix}$$

such that

$$\beta_{i,k} = \sum_{j=1}^{p} \gamma_{ij}\alpha_{jk} \qquad (i = 1, 2, \ldots, p, \quad k = 1, 2, \ldots, n).$$

Using matrix multiplication, we can write these equations as $B = CA$.

745. Show that problem 743 is a special case of problem 744.

746. Prove that if the rank of a homogeneous system of linear equations is less by unity than the number of unknowns, then any two solutions of the system are proportional, that is, differ by only numerical factor (which may be zero).

747. Using the theory of homogeneous systems of linear equations, solve problem 509, that is, prove that if a determinant D of order $n > 1$ is equal to zero, then the cofactors of the corresponding elements of any two rows (or columns) are proportional.

***748.** Prove that if the number of equations in a homogeneous system of linear equations is less by unity than the number of unknowns, then for a solution we can take a system of minors obtained from the matrix of coefficients by a sequential crossing out of the 1st, 2nd, etc. columns (the minors are taken with alternating signs).

Furthermore, show that if this solution is not a zero solution, then any solution is obtained from it by multiplying by some scalar.

Use the result of the preceding problem to find a particular solution and the general solution of each of the following systems of equations:

749. $5x_1 + 3x_2 + 4x_3 = 0$,
$6x_1 + 5x_2 + 6x_3 = 0$.

750. $4x_1 - 6x_2 + 5x_3 = 0$,
$6x_1 - 9x_2 + 10x_3 = 0$.

751. $2x_1 + 3x_2 + 5x_3 + 6x_4 = 0$,
$3x_1 + 4x_2 + 6x_3 + 7x_4 = 0$,
$3x_1 + x_2 + x_3 + 4x_4 = 0$.

752. $8x_1 - 5x_2 - 6x_3 + 3x_4 = 0$,
$4x_1 - x_2 - 3x_3 + 2x_4 = 0$,
$12x_1 - 7x_2 - 9x_3 + 5x_4 = 0$.

753. Prove that for a system of linear equations with the number of equations one more than the number of unknowns to be consistent, it is necessary (but not sufficient) that the determinant made up of all coefficients of the unknowns and the constant terms be zero. Show that this condition will also be sufficient if the rank of the matrix of the coefficients is equal to the number of the unknowns

754. Given: the system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \ldots, s)$$

and two solutions of this system ($\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$) and also the number λ. Find a system of linear equations with the same coefficients of the unknowns as in the given system and such that it has for its solution

(a) the sum of the given solutions:

$$\alpha_1 + \beta_1, \quad \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n$$

or

(b) the product of the first of the given solutions into the number λ:

$$\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n.$$

755. Find the necessary and sufficient conditions for either the sum of two solutions or the product of one solution by a number $\lambda \neq 1$ to again be a solution of the same system of linear equations.

756. Under what conditions will a given linear combination of any solutions of a given nonhomogeneous system of linear equations again be a solution of the system?

757. What values can the unknowns assume in any solutions of a consistent system of linear equations if the columns of the coefficients of the unknowns, except the first, and also the column of constant terms differ pairwise solely in numerical factors?

758. Under what conditions does the unknown x_k have the same value in any solution of a consistent system of linear equations?

759. Find the necessary and sufficient conditions for the kth unknown to be zero in any solution of a consistent system of linear equations.

760. In the general solution of the system of equations

$$\begin{aligned} y + az + bt &= 0, \\ -x + cz + dt &= 0, \\ ax + cy - et &= 0, \\ bx + dy + ez &= 0, \end{aligned}$$

under what conditions can z and t be taken for the free unknowns?

761. How many independent conditions must be fulfilled for a system of s linear equations in n unknowns to be consistent and to have r independent equations from which the remaining equations are derived?

762. Under what conditions does the following system of equations

$$\begin{aligned} x &= by + cz + du + ev, \\ y &= cz + du + ev + ax, \\ z &= du + ev + ax + by, \\ u &= ev + ax + by + cz, \\ v &= ax + by + cz + du \end{aligned}$$

have a nonzero solution?

***763.** Under what conditions does the system of linear equations with real coefficients

$$\begin{aligned} \lambda x + ay + bz + ct &= 0, \\ -ax + \lambda y + hz - gt &= 0, \\ -bx - hy + \lambda z + ft &= 0, \\ -cx + gy - fz + \lambda t &= 0 \end{aligned}$$

have a nontrivial solution?

Use the theory of linear equations to solve the following problems (assume only Cartesian rectangular systems of coordinates).

764. Find the necessary and sufficient conditions for the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) to be collinear.

765. Write the equation of a straight line passing through the two points (x_1, y_1), (x_2, y_2).

766. Find the necessary and sufficient conditions for the three straight lines

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_3x + b_3y + c_3 &= 0 \end{aligned}$$

to be concurrent.

767. Find the necessary and sufficient conditions for the following n points of a plane to be collinear (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n).

768. Find the necessary and sufficient conditions for the following n straight lines in a plane to be concurrent:

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ &\cdots\cdots\cdots\cdots \\ a_nx + b_ny + c_n &= 0. \end{aligned}$$

769. Find the necessary and sufficient conditions for four points in a plane, (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), not lying on a single straight line, to lie on the circumference of a circle.

770. Write down the equation of a circle passing through three points (x_1, y_1), (x_2, y_2), (x_3, y_3) that do not lie on one straight line.

771. Write down the equation of a circle passing through three points (1, 2), (1, —2), (0, —1) and find the centre and radius of the circle.

***772.** Prove that a circle passing through three points with rational coordinates has a centre in a point that also has rational coordinates.

773. Write down the equation of a second-degree curve passing through five points:

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

774. Find the equation of a second-degree curve (and determine the shape of the curve) that passes through the following five points:

$$(3, 0),\ (-3, 0),\ \left(5, 6\frac{2}{3}\right),\ \left(5, -6\frac{2}{3}\right),\ \left(-5, -6\frac{2}{3}\right).$$

775. Write down the equation of a second-degree curve passing through the following five points (also determine the position and dimensions of the curve):

$$(0, 1),\ (\pm 2, 0),\ (\pm 1, -1).$$

776. Find the necessary and sufficient condition for four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) to lie in one plane.

777. Write down the equation of a plane that passes through the following three points:

$$(1, 1, 1),\quad (2, 3, -1),\quad (3, -1, -1).$$

778. Find the necessary and sufficient conditions for the four planes

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0,\\ a_2x + b_2y + c_2z + d_2 &= 0,\\ a_3x + b_3y + c_3z + d_3 &= 0,\\ a_4x + b_4y + c_4z + d_4 &= 0 \end{aligned}$$

to be concurrent.

779. Find the necessary and sufficient conditions for n planes $a_ix + b_iy + c_iz + d_i = 0$ $(i = 1, 2, \ldots, n)$ to pass through one straight line but not to merge into a single plane.

780. Write down the equation of a sphere that passes through four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) not lying in a single plane.

781. Write down the equation and find the centre and radius of a sphere that passes through the points (1, 1, 1), (1, 1, −1), (1, −1, 1), (−1, 0, 0).

782. What kind of system of linear equations specifies three distinct concurrent straight lines in a plane?

783. What kind of system of linear equations specifies three straight lines in a plane that form a triangle?

784. What system of linear equations specifies three planes in space that do not have any common points but intersect in pairs?

785. What system of linear equations specifies four planes in space that form a tetrahedron?

786. Give a geometrical interpretation of a system of four linear equations in three unknowns in which the ranks of all matrices made up of the coefficients of the unknowns of three equations and the rank of the augmented matrix are equal to three.

787. Consider all possible cases encountered in solving systems of linear equations involving two and three unknowns, and in each instance give a geometrical interpretation of the given system of equations.

Chapter III

Matrices and Quadratic Forms

Sec. 12. Operations Involving Matrices

Compute the following matrix products:

788. $\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$.

789. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

790. $\begin{pmatrix} 1 & -3 & 2 \\ 3 & -4 & 1 \\ 2 & -5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 & 6 \\ 1 & 2 & 5 \\ 1 & 3 & 2 \end{pmatrix}$.

791. $\begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{pmatrix}$.

792. $\begin{pmatrix} 2 & -1 & 3 & -4 \\ 3 & -2 & 4 & -3 \\ 5 & -3 & -2 & 1 \\ 3 & -3 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 7 & 8 & 6 & 9 \\ 5 & 7 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 2 & 1 & 1 & 2 \end{pmatrix}$.

793. $\begin{pmatrix} 5 & 7 & -3 & -4 \\ 7 & 6 & -4 & -5 \\ 6 & 4 & -3 & -2 \\ 8 & 5 & -6 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$.

794. $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot \begin{pmatrix} 9 & -6 \\ 6 & -4 \end{pmatrix}$.

795. $\begin{pmatrix} 5 & 2 & -2 & 3 \\ 6 & 4 & -3 & 5 \\ 9 & 2 & -3 & 4 \\ 7 & 6 & -4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 & 2 \\ -1 & -5 & 3 & 11 \\ 16 & 24 & 8 & -8 \\ 8 & 16 & 0 & -16 \end{pmatrix}$.

796. $\begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix} \cdot \begin{pmatrix} -28 & 93 \\ 38 & -126 \end{pmatrix} \cdot \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$

797. $\begin{pmatrix} 0 & 2 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 70 & 34 & -107 \\ 52 & 26 & -68 \\ 101 & 50 & -140 \end{pmatrix} \times$
$\times \begin{pmatrix} 27 & -18 & 10 \\ -46 & 31 & -17 \\ 3 & 2 & 1 \end{pmatrix}$

798. $\begin{pmatrix} 1 & 1 & 1 & -1 \\ -5 & -3 & -4 & 4 \\ 5 & 1 & 4 & -3 \\ -16 & -11 & -15 & 14 \end{pmatrix} \cdot \begin{pmatrix} 7 & -2 & 3 & 4 \\ 11 & 0 & 3 & 4 \\ 5 & 4 & 3 & 0 \\ 22 & 2 & 9 & 8 \end{pmatrix}$

Evaluate the following expressions:

799. $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}^3$.

800. $\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}^5$.

801. $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}^n$.

802. $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n$.

803. $\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}^k$, where the zeros mean that all elements outside the main diagonal are zero.

804. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$.

805. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$.

806. $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}^3$.

807. $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}^{n-1}$.

(the order of the given matrix is n).

808. Compute $\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix}^5$ by using the equation

$$\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}.$$

809. Compute $\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}^6$ by using the equation

$$\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -5 & 4 \end{pmatrix}.$$

810. Prove that if for the matrices A and B both products AB and BA exist, where $AB = BA$, then A and B are square matrices and have the same order.

811. How will the product AB of matrices A and B change if we

(a) interchange the ith and jth rows of A?

(b) add the jth row multiplied by the scalar c to the ith row of A?

(c) interchange the ith and jth columns of B?

(d) add to the ith column of B the jth column multiplied by the scalar c?

812. Using the preceding problem and the invariability of the rank under elementary transformations (see problem 615), prove that the rank of a product of two matrices does not exceed the rank of each of the factors.

813. Prove that the rank of a product of several matrices does not exceed the rank of each of the matrices being multiplied.

814. The *trace* of a square matrix is the sum of the elements on the principal diagonal. Prove that the trace of AB is equal to the trace of BA.

815. Prove that if A and B are square matrices of one and the same order, $AB \neq BA$, then

(a) $(A + B)^2 \neq A^2 + 2AB + B^2$,

(b) $(A + B)(A - B) \neq A^2 - B^2$.

816. Prove that if $AB = BA$, then

$$(A+B)^n = A^n + nA^{n-1}B + \frac{n(n-1)}{2} A^{n-2}B^2 + \ldots + B^n.$$

Here, A and B are square matrices of the same order.

817. Prove that any square matrix A may be represented uniquely in the form $A = B + C$, where B is a symmetric matrix and C is a skew-symmetric matrix.

818. Two matrices A and B are said to *commute* (or to be *commuting* matrices) if $AB = BA$. The square matrix A is said to be a *scalar* matrix if all the elements outside the principal diagonal are zero, while the elements on the principal diagonal are equal, that is, if $A = cE$, where c is a scalar and E is the unit matrix. Prove the following assertion: for a square matrix A to be permutable (i.e. for it to commute with all square matrices of the same order) it is necessary and sufficient for A to be a scalar matrix.

819. A square matrix is said to be *diagonal* if all its elements outside the principal diagonal are zero. Prove the following assertion: for a square matrix A to commute with all diagonal matrices, it is necessary and sufficient for A to be a diagonal matrix.

820. Prove that if A is a diagonal matrix and all the elements of the principal diagonal are distinct, then any matrix that commutes with A is also a diagonal matrix.

821. Prove that premultiplication of matrix A by a diagonal matrix $B = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ brings about a multiplication of the rows of A by $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, while a postmultiplication of A by B brings about a similar change in the columns.

Find all matrices that commute with the following matrices:

822. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

823. $\begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}$.

824. $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

825. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

826. Find all the numbers c which, when multiplied by the nonsingular matrix A, do not change the determinant of the matrix.

827. Find the value of the polynomial $f(x) = 3x^2 - 2x + 5$ of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -5 & 2 \end{pmatrix}.$$

828. Find the value of the polynomial $f(x) = x^3 - 7x^2 + 13x - 5$ of the matrix

$$A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 3 & -1 \\ 2 & 2 & -1 \end{pmatrix}.$$

829. Prove that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the equation

$$x^2 - (a + d)x + ad - bc = 0.$$

*830. Prove that for any square matrix A there exists a polynomial $f(x)$ that is different from zero and is such that $f(A) = 0$, and all such polynomials are divisible by one of them, which is uniquely defined by the condition that the leading coefficient is equal to unity (it is termed the minimal polynomial of the matrix A).

*831. Prove that the equality $AB - BA = E$ does not hold no matter what the matrices A and B.

832. Find all second-order matrices whose squares are equal to the zero matrix.

*833. Let A be a second-order matrix and k an integer exceeding two. Prove that $A^k = 0$ if and only if $A^2 = 0$.

834. Find all second-order matrices whose squares are equal to the unit matrix.

835. Investigate the equation $AX = 0$, where A is a given second-order matrix and X is the desired second-order matrix.

Find the inverses of the following matrices:

836. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

837. $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$.

838. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

839. $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

840. $\begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}$.

841. $\begin{pmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{pmatrix}$.

842. $\begin{pmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{pmatrix}$.

843. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

844. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

845. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{pmatrix}$.

846. $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

847. $\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$

848. $\begin{pmatrix} 1 & a & a^2 & a^3 & \dots & a^n \\ 0 & 1 & a & a^2 & \dots & a^{n-1} \\ 0 & 0 & 1 & a & \dots & a^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

849. $\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ a & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & 1 \end{pmatrix}$

(the order of the matrix is $n + 1$).

850. $\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 0 & 1 & 2 & 3 & \dots & n-2 & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-3 & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

851. $\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}$.

852. $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$.

853. $\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$.

854. $\begin{pmatrix} 1+a & 1 & 1 & \dots & 1 \\ 1 & 1+a & 1 & \dots & 1 \\ 1 & 1 & 1+a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1+a \end{pmatrix}$ (the order of the matrix is n).

855. $\begin{pmatrix} 1+a_1 & 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & 1 & \dots & 1 \\ 1 & 1 & 1+a_3 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1+a_n \end{pmatrix}$.

856. Show that computing the inverse of a given nth-order matrix can be reduced to solving n systems of linear equations, each system containing n equations in n unknowns and having a matrix A for the matrix of coefficients of the unknowns.

Using the method of problem 856, find the inverse of the following matrices:

857. $\begin{pmatrix} 3 & 3 & -4 & -3 \\ 0 & 6 & 1 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 3 & 3 & 2 \end{pmatrix}$.

*858. $\begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & 1 & 2 & \dots & n-2 & n-1 \\ n-1 & n & 1 & \dots & n-3 & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 3 & 4 & \dots & n & 1 \end{pmatrix}$.

859. $\begin{pmatrix} a & a+h & a+2h & \dots & a+(n-2)h & a+(n-1)h \\ a+(n-1)h & a & a+h & \dots & a+(n-3)h & a+(n-2)h \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a+h & a+2h & a+3h & \dots & a+(n-1)h & a \end{pmatrix}$.

*860. $\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^6 & \dots & \varepsilon^{2(n-1)} \\ 1 & \varepsilon^3 & \varepsilon^6 & \varepsilon^9 & \dots & \varepsilon^{3(n-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \varepsilon^{3(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{pmatrix}$,

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Solve the following matrix equations:

861. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$.

862. $X \cdot \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$.

863. $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$.

864. $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}$.

865. $X \cdot \begin{pmatrix} 5 & 3 & 1 \\ 1 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 3 & 0 \\ -5 & 9 & 0 \\ -2 & 15 & 0 \end{pmatrix}$.

866. $\begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{pmatrix}$.

867. $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$.

868. $X \cdot \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 9 & 18 \end{pmatrix}$.

869. $\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

870. $\begin{pmatrix} 3 & -1 & 2 \\ 4 & -3 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 9 & 7 \\ 1 & 11 & 7 \\ 7 & 5 & 7 \end{pmatrix}$.

871. $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

872. How will the inverse matrix A^{-1} change if in the given matrix A we

(a) interchange the ith and jth rows?

(b) multiply the ith row by a nonzero scalar c?

(c) add to the ith row the jth row multiplied by a scalar c, or do the same with respect to the columns?

873. An integral square matrix is said to be *unimodular* if its determinant is equal to ± 1. Prove that an integral matrix has an integral inverse if and only if the given matrix is unimodular.

874. Prove that the matrix equation $AX = B$ is solvable if and only if the rank of A is equal to the rank of the matrix (A, B) obtained from A by adjoining B on the right.

875. Show that the matrix equation $AX = 0$, where A is a square matrix, has a nonzero solution if and only if $|A| = 0$.

876. Let A and B be nonsingular matrices of one and the same order. Show that the following four equalities are equivalent:

$$AB = BA, \quad AB^{-1} = B^{-1}A, \quad A^{-1}B = BA^{-1},$$

$$A^{-1}B^{-1} = B^{-1}A^{-1}.$$

877. Let A be a square matrix and let $f(x)$ and $g(x)$ be any polynomials. Show that the matrices $f(A)$ and $g(A)$ commute, that is, $f(A)\, g(A) = g(A)\, f(A)$.

878. Let A be a square matrix and let $r(x) = \frac{f(x)}{g(x)}$ be a rational function of x. Show that the value $r(A)$ of the

function $r(x)$ for $x = A$ is defined uniquely if and only if $|g(A)| \neq 0$.

879. Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} E_k & U \\ 0 & E_l \end{pmatrix}$, where E_k and E_l are unit matrices of orders k and l respectively, and U is an arbitrary matrix (k, l) (that is, a matrix made up of k rows and l columns), while all other elements are zero.

880. *A kth skew series of order n* is the term used for a square matrix $H_k = (h_{ij})$ of order n whose elements are defined by

$$h_{ij} = \begin{cases} 1 \text{ when } j - i = k \\ 0 \text{ when } j - i \neq k \end{cases} \quad (k = \pm 1, \pm 2, \ldots, \pm (n-1)).$$

Show that $H_1^k = H_k$, $H_{-1}^k = H_{-k}$ if $k = 1, 2, \ldots, n-1$; $H_1^k = H_{-1}^k = 0$ if $k \geqslant n$.

881. How does the matrix A change when pre- and post-multiplied by the matrix H_1 or H_{-1} of the preceding problem?

882. Show that the operation of transposing a matrix has the following properties:

$$\text{(a)} \quad (A + B)' = A' + B', \quad \text{(b)} \quad (AB)' = B'A',$$
$$\text{(c)} \quad (cA)' = cA', \quad \text{(d)} \quad (A^{-1})' = (A')^{-1},$$

where c is a scalar and A and B are matrices.

883. Show that if A and B are symmetric square matrices of the same order, then the matrix $C = ABAB \ldots ABA$ is symmetric.

884. Show that

(a) the inverse of a nonsingular symmetric matrix is symmetric;

(b) the inverse of a nonsingular skew-symmetric matrix is skew-symmetric.

885. Show that for any matrix B a matrix $A = BB'$ is symmetric.

886. Suppose $A^* = \overline{A}'$ is a matrix obtained from A by transposing and replacing all elements by complex conjugate numbers. Show that

$$\text{(a)} \quad (A + B)^* = A^* + B^*, \quad \text{(b)} \quad (AB)^* = B^*A^*,$$
$$\text{(c)} \quad (cA)^* = \overline{c}A^*, \quad \text{(d)} \quad (A^{-1})^* = (A^*)^{-1},$$

where c is a scalar and A and B are matrices that can be subjected to the indicated operations.

887. A matrix A is said to be *Hermitian* if $A^* = A$. Show that for any matrix B with complex or real elements the matrix $A = B \cdot B^*$ is Hermitian.

888. Show that the product of two symmetric matrices is a symmetric matrix if and only if the given matrices commute.

889. Show that the product of two skew-symmetric matrices is a symmetric matrix if and only if the given matrices commute.

***890.** Prove that the product of two skew-symmetric matrices A and B is skew-symmetric if and only if $AB = -BA$.

Give examples of skew-symmetric matrices that satisfy the condition $AB = -BA$.

891. A square matrix $A = (a_{ij})$ of order n is said to be *orthogonal* if $AA' = E$, where E is the unit matrix. Show that for a square matrix A to be orthogonal it is necessary and sufficient that any one of the following conditions holds:

(a) the columns of A form an orthonormal system, that is,

$$\sum_{k=1}^{n} a_{ki}a_{kj} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta, which equals 1 for $i = j$ and 0 when $i \neq j$;

(b) the rows of A form an orthonormal system, that is,

$$\sum_{k=1}^{n} a_{ik}a_{jk} = \delta_{ij}.$$

892. A square matrix $A = (a_{ij})$ of order n with real or complex elements is said to be *unitary* if $AA^* = E$ (the meaning of A^* is the same as in problem 886). Show that A is unitary if and only if any one of the following conditions holds:

(a) $\sum_{k=1}^{n} a_{ki}\bar{a}_{kj} = \delta_{ij}$

(b) $\sum_{k=1}^{n} a_{ik}\bar{a}_{jk} = \delta_{ij}$

(δ_{ij} is the Kronecker delta).

893. Prove that the determinant of an orthogonal matrix is equal to ± 1.

894. Prove that the determinant of a unitary matrix is in modulus equal to unity.

895. Prove that if an orthogonal matrix A has square submatrices $A_1, A_2, \ldots, A_s$ on the principal diagonal and zeroes on one side of the submatrices, then all elements on the other side are zero and all the matrices $A_1, A_2, \ldots, A_s$ are orthogonal.

896. Prove that for a square matrix A to be orthogonal it is necessary and sufficient that its determinant be equal to ± 1 and that each element be equal to its cofactor taken with its sign if $|A| = 1$ and with sign reversed if $|A| = -1$.

***897.** Prove that a real square matrix A of order $n \geqslant 3$ is orthogonal if each element is equal to its cofactor and at least one of the elements is nonzero.

***898.** Prove that a real square matrix A of order $n \geqslant 3$ is orthogonal if each element is equal to its cofactor with sign reversed and at least one of the elements is different from zero.

***899.** Prove that the sum of the squares of all second-order minors lying in two rows (or columns) of an orthogonal matrix is equal to unity.

***900.** Prove that the sum of the square of the moduli of all second-order minors lying in two rows (or columns) of a unitary matrix is equal to unity.

***901.** Prove that the sum of the squares of all kth-order minors lying in any k rows (or columns) of an orthogonal matrix is equal to unity.

***902.** Prove that the sum of the squares of the moduli of all kth-order minors lying in any k rows (or columns) of a unitary matrix is equal to unity.

***903.** Prove that the minor of any order of an orthogonal matrix A is equal to its cofactor taken with its sign if $|A| = 1$ and with sign reversed if $|A| = -1$.

***904.** Let A be a unitary matrix and M be its minor of any order, and let M_A be the cofactor of M in A. Prove that $M_A = |A| \cdot \overline{M}$, where $\overline{M}$ is the conjugate of M.

905. Under what conditions is a diagonal matrix orthogonal?

906. Under what conditions is a diagonal matrix unitary?

907. Verify that any one of the following three properties of a square matrix follows from the other two: reality, orthogonality, unitarity.

908. A square matrix I is said to be *involutory* if $I^2 = E$. Show that each of the following three properties of a square matrix follows from the other two: symmetry, orthogonality, involutoriness.

909. Verify that the matrices

(a) $\begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$; (b) $\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$

possess all three properties of the preceding problem.

910. A square matrix P is said to be *idempotent* if $P^2 = P$. Show that if P is idempotent, then $I = 2P - E$ is involutory, and conversely, if I is involutory, then $P = 1/2\,(I + E)$ is idempotent.

911. Prove that

(a) the product of two orthogonal matrices is an orthogonal matrix;

(b) the inverse of an orthogonal matrix is orthogonal.

912. Prove that

(a) the product of two unitary matrices is a unitary matrix;

(b) the inverse of a unitary matrix is unitary.

***913.** Denote by $A\begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{pmatrix}$ the minor of matrix A lying at the intersection of rows labelled $i_1, i_2, \ldots, i_p$ and columns labelled $j_1, j_2, \ldots, j_p$.

Prove the validity of the following expression of the minors of the product $C = AB$ of two matrices in terms of the minors of the matrices being multiplied together:

$$C\begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{pmatrix} = \sum_{1 \leqslant k_1 < k_2 < \ldots < k_p \leqslant n} A\begin{pmatrix} i_1, i_2, \ldots i_p \\ k_1, k_2, \ldots k_p \end{pmatrix} B\begin{pmatrix} k_1, k_2, \ldots, k_p \\ j_1, j_2, \ldots, j_p \end{pmatrix},$$

$$(i_1 < i_2 < \ldots < i_p;\ j_1 < j_2 < \ldots < j_p),$$

if p does not exceed the number of columns of matrix A or the number of rows in B. Otherwise all minors of order p of matrix C are equal to zero.

914. Using the preceding problem, show that the rank of a product of two matrices does not exceed the rank of each of the factors.

***915.** Prove that pre- or postmultiplication of A by a nonsingular matrix does not alter the rank of the matrix.

916. The principal minor of matrix A is the minor at the intersection of rows and columns with the same number labels. Show that if the elements of matrix B are real, then all principal minors of the matrix $A = BB'$ are nonnegative.

917. Show that for any matrix B with complex or real elements, all principal minors of matrix $A = BB^*$ are nonnegative. Here, $B^* = \bar{B}'$.

918. Show that if, using the notation of the preceding problem, $A = BB^*$, then the rank of A is equal to the rank of B.

919. Prove that the sum of the principal kth-order minors of the matrix AA' is equal to the sum of the squares of all kth-order minors of the matrix A.

***920.** Prove that for any square matrices A and B of order n the sum of all principal minors of a given order k ($1 \leqslant \leqslant k \leqslant n$) for matrices AB and BA is the same.

***921.** Let A be a real matrix of order n and let B and C be matrices made up of the first k and last $n - k$ columns of A. Prove that $|A|^2 \leqslant |B'B| \cdot |C'C|$.

***922.** Let $A = (B, C)$ be a real matrix (the meaning of the symbol (B, C) is given in problem 874). Prove that $|A'A| \leqslant |B'B| \cdot |C'C|$.

***923.** Let $A = (a_{ij})$ be a square real matrix of order n. Prove the Hadamard inequality

$$|A|^2 \leqslant \prod_{k=1}^{n} \sum_{i=1}^{n} a_{ik}^2.$$

***924.** Prove that for any real rectangular matrix $A = (a_{ij})$ with n rows and m columns, the following inequality holds:

$$|A'A| \leqslant \prod_{k=1}^{m} \sum_{i=1}^{n} a_{ik}^2.$$

*925. Let $A = (B, C)$ be a matrix with complex elements. Prove that $|A^* \cdot A| \leqslant |B^* \cdot B| \cdot |C^* \cdot C|$.

*926. Let $A = (a_{ij})$ be a square matrix of order n with complex elements that do not exceed a number M in absolute value. Prove that the modulus of the determinant $|A|$ does not exceed $M^n \cdot n^{\frac{n}{2}}$, the estimate being exact.

*927. Show that each elementary transformation of a matrix A, that is, a transformation of one of the following types:

(a) the transposition of two rows (or columns);

(b) multiplication of a row (or a column) by a number c different from zero;

(c) adding to one row (or one column) another row (or column) multiplied by any scalar c, may be obtained by multiplying the matrix A by some nonsingular matrix P on the left for transforming rows and on the right for transforming columns.

Indicate the type of such matrices.

*928. A square matrix is said to be *triangular* if all the elements on one side of the principal diagonal are equal to zero. Show that any square matrix can be represented in the form of a product of several triangular matrices.

*929. Show that any matrix A of rank r can be expressed as a product $A = PRQ$, where P and Q are nonsingular matrices and R is a rectangular matrix of the same dimensions as A on the principal diagonal of which the first r elements are equal to unity and all the other elements are zero.

*930. Let A be a matrix of size $m \times n$ and of rank r, let $P = (p_{ij})$ be a matrix of size $s \times m$, in which $p_{11} = p_{22} = \ldots = p_{kk} = 1$ and all other elements are zeros, and let $Q = (q_{ij})$ be a matrix of size $n \times t$ in which $q_{11} = q_{22} = \ldots = q_{ll} = 1$ and all other elements are zeros. Prove the following inequalities:

(a) the rank of $PA \geqslant k + r - m$,

(b) the rank of $AQ \geqslant l + r - n$,

(c) the rank of $PAQ \geqslant k + l + r - m - n$.

*931. Denote the rank of the matrix A by r_A. Prove that for the rank of the product AB of two square matrices A and B of order n we have the following inequality:

$r_A + r_B - n \leqslant r_{AB} \leqslant r_A, r_B$ (Sylvester's inequality).

932. Show that for the rank of the product AB of rectangular matrices A and B the Sylvester inequality of the preceding problem holds provided that n denotes the number of columns of A and the number of rows in B.

***933.** Show that it is possible, via elementary transformations of rows alone (or columns alone), to reduce any nonsingular matrix A to the unit matrix E. If the elementary operations performed on A are applied in the same order (sequence) to the unit matrix E, then we obtain A^{-1}, the inverse of A.

Using the device of the preceding problem, find the inverse matrices of the following matrices (for convenience of computation, on the right adjoin to the given matrix A a unit matrix and perform elementary transformations of rows (which reduce A to E) on the rows of the entire matrix):

934. $\begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 8 & 0 & -4 \\ 2 & 2 & -4 & -3 \\ 3 & 8 & -1 & -6 \end{pmatrix}$.

935. $\begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{pmatrix}$.

936. $\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & 4 \\ 2 & 7 & 6 & -1 \\ 1 & 2 & 2 & -1 \end{pmatrix}$.

937. Using the method of problem 933, find the inverses of the matrices of problems 844, 846, 847, 848, 849, 850.

***938.** Prove the following statement:

for the matrix A made up of m rows and n columns to have a rank of unity, it is necessary and sufficient for A to be represented as $A = BC$, where B is a nonzero column of length m and C is a nonzero row of length n.

***939.** Prove the following assertion:

for the matrix A consisting of m rows and n columns to have rank r, it is necessary and sufficient that A be represented as $A = BC$, where B is a matrix of m rows and r linearly independent columns, and C is a matrix of r linearly independent rows and n columns.

940. Show that if A and B are square matrices of order n and $AB = 0$, then $r_A + r_B \leqslant n$; note that for any given matrix A the matrix B can be chosen so that $r_A + r_B = k$, where k is any integer satisfying the condition $r_A \leqslant k \leqslant n$.

*941. Show that if A is a square matrix of order n for which $A^2 = E$, then $r_{E+A} + r_{E-A} = n$.

942. Two integer matrices are said to be equivalent if it is possible to pass from one to another via integer elementary transformations, that is, transformations of the following types:

(a) the transposition of two rows;

(b) multiplication of a row by -1;

(c) the addition to one row of another row multiplied by an integer c, and similar transformations for the columns. Prove that the matrices A and B are equivalent if and only if $B = PAQ$, where P and Q are square integer unimodular matrices.

*943. A rectangular integer matrix A is said to be *normal* if its elements $a_{11}, a_{22}, \ldots, a_{rr}$ are positive, a_{ii} is divisible by $a_{i-1,\, i-1}$ $(i = 2, 3, \ldots, r)$, and all other elements are zero. Show that each integer matrix is equivalent to one and only one normal matrix; in other words, each class of equivalent integer matrices contains a normal matrix and there is only one such normal matrix.

*944. Prove that each nonsingular integer matrix A may be represented as $A = PR$, where P is a unimodular integer matrix and R is a triangular integer matrix whose elements on the main diagonal are positive, those below the main diagonal are zero, and those above the main diagonal are nonnegative and less than the elements of the main diagonal of the given column; also prove that that representation is unique.

*945. Prove that a square matrix A of order n and rank r may be represented in the form $A = PR$, where P is a nonsingular matrix and R is a triangular matrix in which the first r elements of the main diagonal are equal to unity, and all elements below the main diagonal and all elements of the last $n - r$ rows are zero.

946. A square matrix is said to be an *upper* (or *lower*) triangular matrix if all the elements below (or, respectively, above) the main diagonal are zero. Show that the following operations applied to an upper (or lower) triangular matrix lead to an upper (or lower) triangular matrix: the addition of two matrices, the multiplication of a matrix by a scalar, the multiplication of two matrices, and the transition to an inverse matrix for a nonsingular matrix.

947. A square matrix is said to be *nilpotent* if when powered it is equal to zero. The smallest possible integer k for which $A^k = 0$ is called the *exponent of nilpotency* of the matrix A. Show that a triangular matrix is nilpotent if and only if all elements of the main diagonal are zero and the exponent of nilpotency of the triangular matrix does not exceed the order of the matrix.

948. Show that the inverse matrix $B = (b_{ik})$ of the upper (or lower) triangular nonsingular matrix $A = (a_{ik})$ of order n is again an upper (or lower) triangular matrix, and the elements of the main diagonal of B are given by the equations $b_{ii} = \frac{1}{a_{ii}}$ $(i = 1, 2, \ldots, n)$, whereas the remaining elements are found from the following recurrence relations:

(a) for elements of the ith row of the upper triangular matrix:

$$b_{ik} = \frac{-\sum_{j=i}^{k-1} b_{ij}a_{jk}}{a_{kk}} \quad (k = i+1, i+2, \ldots, n);$$

(b) for elements of the kth column of the lower triangular matrix:

$$b_{ik} = \frac{-\sum_{j=k}^{i-1} a_{ij}b_{jk}}{a_{ii}} \quad (i = k+1, k+2, \ldots, n).$$

These formulas are convenient to use when computing matrices inverse to triangular matrices.

***949.** Let A be a square matrix of order n and rank r, and let

$$d_k = A\begin{pmatrix} 1, 2, \ldots, k \\ 1, 2, \ldots, k \end{pmatrix} \neq 0 \quad (k = 1, 2, \ldots, r). \qquad (1)$$

Prove that under these conditions the matrix A may be represented as the product

$$A = BC, \qquad (2)$$

where $B = (b_{ij})$ is the lower and $C = (c_{ij})$ the upper triangular matrix (the definition of upper and lower triangular matrices is given in problem 946).

The first r diagonal elements of the matrices B and C may be given any values that satisfy the conditions

$$b_{kk}c_{kk} = \frac{d_k}{d_{k-1}} \quad (k = 1, 2, \ldots, r;\ d_0 = 1). \tag{3}$$

Specifying the first r diagonal elements of the matrices B and C uniquely defines the remaining elements of the first r columns of B and the first r rows of C; these elements are given by the following formulas:

$$b_{ik} = b_{kk}\frac{A\begin{pmatrix}1, 2, \ldots, k-1, i\\ 1, 2, \ldots, k-1, k\end{pmatrix}}{d_k}, \tag{4}$$

$$c_{ki} = c_{kk}\frac{A\begin{pmatrix}1, 2, \ldots, k-1, k\\ 1, 2, \ldots, k-1, i\end{pmatrix}}{d_k}$$

$$(i = k+1, k+2, \ldots, n;\ k = 1, 2, \ldots, r).$$

In the case of $r < n$, all elements in the last $n - r$ columns of B may be put equal to zero, or in the last $n - r$ rows of C the elements may be arbitrary or, contrariwise, the elements in the last $n - r$ columns of B may be arbitrary and all elements in the last $n - r$ rows of C may be put equal to zero.

The arbitrary elements will not disrupt equation (2). They may be chosen so as to retain the triangular aspect of matrices B and C.

950. Show that the representation (2) of the preceding problem may be found thus: choose the first r elements on the main diagonal of matrices B and C in any way so that they satisfy conditions (3), and compute the remaining elements of the first r columns of B and the first r rows of C by means of the following recurrence relations:

$$b_{ik} = \frac{a_{ik} - \sum_{j=1}^{k-1} b_{ij}c_{jk}}{c_{kk}},$$

$$(i = k+1, k+2, \ldots, n;\ k = 1, 2, \ldots, r);$$

$$c_{ik} = \frac{a_{ik} - \sum_{j=1}^{i-1} b_{ij}c_{jk}}{b_{ii}},$$

$$(k = i+1, i+2, \ldots, n;\ i = 1, 2, \ldots, r).$$

These formulas permit one to find initially the first column of B and the first row of C and then, knowing the $k-1$ columns of B and the $k-1$ rows of C, to find the kth column of B and the kth row of C.

*951. Prove that any symmetric matrix $A=(a_{ij})$ of order n and rank r that satisfies conditions (1) of problem 949 may be represented in the form $A=BB'$, where B is a lower triangular matrix whose elements in the last $n-r$ columns are zero and the elements of the first r columns are given by the formulas

$$b_{ik}=\frac{A\begin{pmatrix}1, 2, \ldots, k-1, i\\ 1, 2, \ldots, k-1, k\end{pmatrix}}{\sqrt{d_k\cdot d_{k-1}}}$$

$$(i=k, k+1, \ldots, n;\ k=1, 2, \ldots, r).$$

952. A matrix A is said to be *partitioned* (it is also called a *block* matrix) if the elements are split up into rectangular blocks by one or several horizontal and vertical partitions (lines). We will denote these blocks by A_{ij}, where i is the number of the block row and j is the number of the block column. Show that the multiplication of two block matrices reduces to the multiplication of blocks regarded as separate elements if and only if the vertical partition of the first matrix corresponds to the horizontal partition of the second matrix. Namely, if $A=(A_{ij})$ is an $m\times n$ matrix with partitioning of rows into groups of $m_1, m_2, \ldots, m_s$ and of columns into groups of $n_1, n_2, \ldots, n_t$ and $B=(B_{ij})$ is an $n\times p$ matrix with partitioning of rows into groups of $n_1, n_2, \ldots, n_t$ and of columns into groups of $p_1, p_2, \ldots, p_u$, then $AB=C=(C_{ij})$ will also be a block matrix, and we also have

$$C_{ik}=\sum_{j=1}^{t}A_{ij}B_{jk}\qquad (i=1, 2, \ldots, s;\ k=1, 2, \ldots, u).$$

Use this rule for multiplying block matrices to find the blocks of a product of the following matrices for the indicated partitioning into blocks for the factors

$$A=\left(\begin{array}{c|cc}1 & -2 & 3\\ \hline 3 & -1 & 2\\ \hline 4 & -2 & 1\end{array}\right),\qquad B=\left(\begin{array}{c|cc}2 & 3 & 1\\ \hline 1 & 2 & 3\\ 2 & 1 & 3\end{array}\right).$$

953. Show that to multiply two square partitioned matrices it is sufficient (but, as the example of the preceding problem shows, not necessary) for the diagonal blocks to be square and the orders of the appropriate diagonal blocks to be equal.

***954.** Show that for block multiplication of a partitioned matrix into itself to be possible it is necessary and sufficient for all the diagonal blocks to be square matrices.

955. A square block (or partitioned) matrix $A = (A_{ij})$ is said to be a *block-triangular* matrix if all blocks on the main diagonal, that is, $A_{11}, A_{22}, \ldots$, are square, and all the blocks on any one side of the main diagonal are zero. Show that if A and B are two block-triangular matrices with the same orders of the corresponding diagonal blocks and with zeros on any one side of the diagonal, then their product AB is also a block-triangular matrix with the same orders of the diagonal blocks and with zeros on the same side of the diagonal.

956. Show that a block-triangular matrix is nilpotent if and only if all the blocks on the main diagonal are nilpotent (the definition of nilpotency is given in problem 947).

957. Let $A = (A_{ij})$ be a block matrix, and let A_{ij} be a block of size $m_i \times n_j$ ($i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$). Show that adding to the ith block row the jth row multiplied on the left by a rectangular matrix X of size $m_i \times m_j$ may be obtained by premultiplying A by a nonsingular square block matrix P. In exactly the same way, adding to the ith block column the jth column postmultiplied by a rectangular matrix Y of size $n_j \times n_i$ may be obtained by postmultiplying A by a nonsingular square block matrix Q. Find the forms of the matrices B and Q.

***958.** Let $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a partitioned matrix, where A is a nonsingular square matrix of order n. Prove that the rank R is equal to n if and only if $D = CA^{-1}B$.

***959.** Let A be a nonsingular matrix of order n and let B be an $n \times q$ matrix, and C a $p \times n$ matrix. Prove that if the partitioned matrix $R = \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix}$ is reduced to the form $R_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & X \end{pmatrix}$ via a series of elementary transformations of the rows with each transformation either involv-

ing only the first n rows or the addition to some one row with number label exceeding n of one of the first n rows multiplied by a scalar, then $X = CA^{-1}B$.

960. Let A be a nonsingular matrix of order n and let E be a unit matrix of the same order. Prove that if the partitioned matrix $\begin{pmatrix} A & E \\ -E & 0 \end{pmatrix}$ is reduced to the form $\begin{pmatrix} A_1 & B_1 \\ 0 & X \end{pmatrix}$ via the elementary transformations indicated in the preceding problem, then $X = A^{-1}$. Use this device to find the inverse matrix of $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 2 & 7 \\ 3 & 4 & 4 \end{pmatrix}$.

961. Suppose we have the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

with A a nonsingular matrix of coefficients, with B the column of constant terms, and with E a unit matrix of order n.]

Show that if the partitioned matrix $\begin{pmatrix} A & B \\ -E & 0 \end{pmatrix}$ is reduced to the form $\begin{pmatrix} A_1 & B_1 \\ 0 & X \end{pmatrix}$ by the transformations indicated in problem 959, then the column X yields the solution of the given system of equations.

Use this method to solve the following system:

$$\begin{aligned} 3x - y + 2z &= 7, \\ 4x - 3y + 2z &= 4, \\ 2x + y + 3z &= 13. \end{aligned}$$

962. Let A be a nonsingular matrix of order n, B an $n \times p$ matrix, and let E be a unit matrix of order n. Show that if the matrix $\begin{pmatrix} A & B \\ -E & 0 \end{pmatrix}$ is brought to the form $\begin{pmatrix} A_1 & B_1 \\ 0 & X \end{pmatrix}$ by the transformations given in problem 959, then the matrix X yields the solution of the matrix equation $AX = B$.

Use this method to solve the indicated equation if

$$A=\begin{pmatrix}2 & -7\\ 1 & -4\end{pmatrix}, \qquad B=\begin{pmatrix}4 & -5\\ 1 & -4\end{pmatrix}.$$

*963. Suppose all the pairs (i, j) $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, n)$ are numbered in some specific order $\alpha_1, \alpha_2, \ldots, \alpha_{mn}$. The *Kronecker* (or *direct*) product of two square matrices A of order m and B of order n is a matrix $C = A \times B$ of order mn made up of all possible products of the elements of A and B in the appropriate order. Namely, the element of C in the ith row and jth column is defined thus:

$$c_{ij} = a_{i_1 j_1} b_{i_2 j_2}, \quad \text{where} \quad (i_1, i_2) = \alpha_i, \quad (j_1, j_2) = \alpha_j.$$

Prove that

(a) $(A + B) \times C = (A \times C) + (B \times C)$,
(b) $A \times (B + C) = (A \times B) + (A \times C)$,
(c) $(AB) \times (CD) = (A \times C)(B \times D)$.

*964. *The right direct product* of a square matrix A of order m by a square matrix B of order n is a partitioned matrix $A \times {}^{\cdot}B = C = (C_{ij})$, where $C_{ij} = a_{ij}B$ $(i, j = 1, 2, \ldots, m)$. Similarly, the left direct product of the same matrices is a partitioned matrix $A^{\cdot} \times B = D = (D_{ij})$, where $D_{ij} = Ab_{ij}$ $(i, j = 1, 2, \ldots, n)$.

Prove that

(a) both of the introduced products are particular cases of the Kronecker product defined in the preceding problem. Find the numbering order of the pairs (i, j) that yields the right and left direct products;

(b) $A \times {}^{\cdot}B = B^{\cdot} \times A$

(c) $A \times {}^{\cdot}(B \times {}^{\cdot}C) = (A \times B) \times {}^{\cdot}C$;

(d) if E_k is a unit matrix of order k, then $E_m \times {}^{\cdot}E_n = E_n \times {}^{\cdot}E_m = E_{mn}$;

(e) if A and B are nonsingular matrices, then $(A \times {}^{\cdot}B)^{-1} = A^{-1} \times {}^{\cdot}B^{-1}$.

Properties similar to (c), (d) and (e) hold true for the left product.

*965. Using the two preceding problems, prove that if A is a matrix of order m and B is a matrix of order n, then $|A \times B| = |A|^n \times |B|^m$ (see problem 540).

*966. Let $A = (a_{ij})$ be a square matrix of order n. The *adjoint* (or *adjugate*) of A is the matrix $\hat{A} = (\hat{a}_{ij})$, where

$\hat{a}_{ij} = A_{ji}$ $(i, j = 1, 2, \ldots, n)$. In other words, the adjoint of A is obtained by taking the transpose of a matrix made up of the cofactors of the elements of A.

Prove that

(a) $A\hat{A} = \hat{A}A = |A|E$, where E is the unit matrix;

(b) $(\hat{\hat{A}}) = |A|^{n-2}A$ for $n > 2$, $(\hat{\hat{A}}) = A$ for $n = 2$.

*967. Show that $(\widehat{AB}) = \hat{B} \cdot \hat{A}$, where $\hat{A}$ is the adjoint matrix of A defined in the preceding problem.

*968. The matrix *associated* with a square matrix A of order n is a matrix $\tilde{A} = (\tilde{a}_{ij})$, where $\tilde{a}_{ij}$ is the minor of the element a_{ij} of matrix A. Prove that

(a) $(\widetilde{AB}) = \tilde{A}\tilde{B}$;

(b) $(\tilde{\tilde{A}}) = |A|^{n-2}A$ for $n > 2$, $(\tilde{\tilde{A}}) = A$ for $n = 2$.

*969. Let $A = (a_{ij})$ be a square matrix of order n and let all combinations of the n numbers $1, 2, \ldots, n$ taken p at a time $(k_1 < k_2 < \ldots < k_p)$ be numbered in some order $\alpha_1, \alpha_2, \ldots, \alpha_N$, where $N = C_n^p$; the pth associated matrix relative to A is the matrix $A_p = (a_{i,j;p})$ made up of appropriately arranged minors of the pth order of matrix A; namely, $a_{i,j;p} = A\begin{pmatrix} i_1, & i_2, & \ldots, & i_p \\ j_1, & j_2, & \ldots, & j_p \end{pmatrix}$, where α_i is the combination $i_1 < i_2 < \ldots < i_p$, and α_j is the combination $j_1 < j_2 < \ldots < j_p$.

Prove that

(a) $(AB)_p = A_pB_p$;

(b) $(E_n)_p = E_N$, where E_n and E_N are unit matrices of order n and order N respectively.

(c) if A is a nonsingular matrix, then $(A^{-1})_p = (A_p)^{-1}$.

970. Find a numbering of the combinations of n numbers $1, 2, \ldots, n$ taken p at a time such that relative to the triangular matrix A the associated matrix A_p defined in the preceding problem is also triangular with zeros on the same side of the diagonal.

*971. Using the properties of associated matrices, prove that if A is a square matrix of order n, then $|A_p| = |A|^{C_{n-1}^{p-1}}$ (see problem 551).

*972. Let A be a nonsingular matrix of order n and let $B = A^{-1}$ be the inverse. Prove that the minors of any order of the inverse can be expressed in terms of the minors of the original matrix in the following manner:

$$B\begin{pmatrix} i_1, & i_2, & \ldots, & i_p \\ k_1, & k_2, & \ldots, & k_p \end{pmatrix} = \frac{(-1)^{\sum_{s=1}^{p}(i_s+k_s)} A\begin{pmatrix} k'_1, & k'_2, & \ldots, & k'_{n-p} \\ i'_1, & i'_2, & \ldots, & i'_{n-p} \end{pmatrix}}{|A|}, \quad (1)$$

where $i_1 < i_2 < \ldots < i_p$ together with $i'_1 < i'_2 < \ldots < i'_{n-p}$ and $k_1 < k_2 < \ldots < k_p$ together with $k'_1 < k'_2 < \ldots < k'_{n-p}$ constitute a complete system of indices $1, 2, \ldots, n$.

*973. Prove that the pth associated matrix A_p (the definition is given in problem 969) relative to the orthogonal matrix A is itself orthogonal.

*974. Prove that the pth associated matrix A_p of the unitary matrix A is itself unitary.

Sec. 13. Polynomial Matrices

Bring the following λ-matrices to normal diagonal form by means of elementary operations:

975. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

976. $\begin{pmatrix} \lambda^2-1 & \lambda+1 \\ \lambda+1 & \lambda^2+2\lambda+1 \end{pmatrix}$.

977. $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda+5 \end{pmatrix}$.

978. $\begin{pmatrix} \lambda^2-1 & 0 \\ 0 & (\lambda-1)^3 \end{pmatrix}$.

979. $\begin{pmatrix} \lambda+1 & \lambda^2+1 & \lambda^2 \\ 3\lambda-1 & 3\lambda^2-1 & \lambda^2+2\lambda \\ \lambda-1 & \lambda^2-1 & \lambda \end{pmatrix}$.

980. $\begin{pmatrix} \lambda^2 & \lambda^2-\lambda & 3\lambda^2 \\ \lambda^2-\lambda & 3\lambda^2-\lambda & \lambda^3+4\lambda^2-3\lambda \\ \lambda^2+\lambda & \lambda^2+\lambda & 3\lambda^2+3\lambda \end{pmatrix}$.

981. $\begin{pmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-2 \end{pmatrix}$.

982. $\begin{pmatrix} \lambda(\lambda+1) & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & (\lambda+1)^2 \end{pmatrix}$.

983. $\begin{pmatrix} 1-\lambda & \lambda^2 & \lambda \\ \lambda & \lambda & -\lambda \\ 1+\lambda^2 & \lambda^2 & -\lambda^2 \end{pmatrix}$.

*984. The term *invariant factors* of a λ-matrix A of order n is used for the polynomials $E_1(\lambda)$, $E_2(\lambda)$, . . ., $E_n(\lambda)$ on the main diagonal in the normal diagonal form of the matrix A. The divisors of the minors of A are the polynomials $D_1(\lambda)$, $D_2(\lambda)$, . . ., $D_n(\lambda)$, where $D_k(\lambda)$ is the greatest common divisor (with leading coefficient unity) of the minors of the kth order of A if not all these minors are equal to zero, and $D_k(\lambda) = 0$ otherwise. Prove that $E_k(\lambda) \not\equiv 0$ and $D_k(\lambda) \not\equiv 0$ for $k = 1, 2, \ldots, r$, where r is the rank of the matrix A, whereas $E_k(\lambda) = D_k(\lambda) = 0$ for $k = r+1, \ldots, n$. Furthermore, show that $E_k(\lambda) = \frac{D_k(\lambda)}{D_{k-1}(\lambda)}$ ($k = 1, 2, \ldots, r$; $D_0 = 1$).

Using the minor divisors defined in problem 984, bring the following λ-matrices to normal diagonal form:

985. $\begin{pmatrix} \lambda(\lambda-1) & 0 & 0 \\ 0 & \lambda(\lambda-2) & 0 \\ 0 & 0 & (\lambda-1)(\lambda-2) \end{pmatrix}$.

986. $\begin{pmatrix} \lambda(\lambda-1) & 0 & 0 \\ 0 & \lambda(\lambda-2) & 0 \\ 0 & 0 & \lambda(\lambda-3) \end{pmatrix}$.

987.

$$\begin{pmatrix} (\lambda-1)(\lambda-2)(\lambda-3) & 0 & 0 & 0 \\ 0 & (\lambda-1)(\lambda-2)(\lambda-4) & 0 & 0 \\ 0 & 0 & (\lambda-1)(\lambda-3)(\lambda-4) & 0 \\ 0 & 0 & 0 & (\lambda-2)(\lambda-3)(\lambda-4) \end{pmatrix}.$$

988. $\begin{pmatrix} a^2cd & 0 & 0 & 0 \\ 0 & b^2cd & 0 & 0 \\ 0 & 0 & abc^2 & 0 \\ 0 & 0 & 0 & abd^2 \end{pmatrix}$,

where a, b, c, d are pairwise relatively prime polynomials in λ.

989. $\begin{pmatrix} f(\lambda) & 0 \\ 0 & g(\lambda) \end{pmatrix}$, where $f(\lambda)$ and $g(\lambda)$ are polynomials in λ.

990. $\begin{pmatrix} 0 & 0 & fg \\ 0 & fh & 0 \\ gh & 0 & 0 \end{pmatrix}$, where f, g, h are polynomials in λ that are pairwise relatively prime and have leading coefficients unity.

991. $\begin{pmatrix} fg & 0 & 0 \\ 0 & fh & 0 \\ 0 & 0 & gh \end{pmatrix}$,

where f, g, h are polynomials in λ with leading coefficients unity, coprime taken all together, but not necessarily pairwise coprime.

992. $\begin{pmatrix} fg & 0 & 0 \\ 0 & fh & 0 \\ 0 & 0 & gh \end{pmatrix}$,

where f, g, h are any polynomials in λ with leading coefficients equal to unity.

993. $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$.

994. $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$.

995. $\begin{pmatrix} \lambda & -1 & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & \lambda & -1 \\ 1 & 2 & 3 & 4 & 5+\lambda \end{pmatrix}$.

996. $\begin{pmatrix} \lambda+\alpha & \beta & 1 & 0 \\ -\beta & \lambda+\alpha & 0 & 1 \\ 0 & 0 & \lambda+\alpha & \beta \\ 0 & 0 & -\beta & \lambda+\alpha \end{pmatrix}$.

997. $\begin{pmatrix} 2\lambda^2-12\lambda+16 & 2-\lambda & 2\lambda^2-12\lambda+17 \\ 0 & 3-\lambda & 0 \\ \lambda^2-6\lambda+7 & 2-\lambda & \lambda^2-6\lambda+8 \end{pmatrix}$.

998. $\begin{pmatrix} 3\lambda^2-5\lambda+2 & 0 & 3\lambda^2-6\lambda+3 \\ 2\lambda^2-3\lambda+1 & \lambda-1 & 2\lambda^2-4\lambda+2 \\ 2\lambda^2-2\lambda & 0 & 2\lambda^2-4\lambda+2 \end{pmatrix}$.

999. $\begin{pmatrix} \lambda & 1 & 1 & \dots & 1 \\ 0 & \lambda & 1 & \dots & 1 \\ 0 & 0 & \lambda & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$.

Determine whether the following matrices are equivalent:

1000.
$$A=\begin{pmatrix} 3\lambda+1 & \lambda & 4\lambda-1 \\ 1-\lambda^2 & \lambda-1 & \lambda-\lambda^2 \\ \lambda^2+\lambda+2 & \lambda & \lambda^2+2\lambda \end{pmatrix};$$
$$B=\begin{pmatrix} \lambda+1 & \lambda-2 & \lambda^2-2\lambda \\ 2\lambda & 2\lambda-3 & \lambda^2-2\lambda \\ -2 & 1 & 1 \end{pmatrix}.$$

1001.
$$A=\begin{pmatrix} \lambda^2+\lambda+1 & 3\lambda-\lambda^2 & 2\lambda^2+\lambda & \lambda^2 \\ \lambda^2+\lambda & 3\lambda-\lambda^2 & 2\lambda^2+\lambda & \lambda^2 \\ \lambda^2-\lambda & 2\lambda-\lambda^2 & 2\lambda^2+\lambda & \lambda^2 \\ \lambda^2 & -\lambda^2 & 2\lambda^2 & \lambda^2 \end{pmatrix};$$
$$B=\begin{pmatrix} 3 & \lambda^2+1 & 3\lambda^2 & \lambda^2 \\ 2 & \lambda^2+1 & 3\lambda^2 & \lambda^2 \\ 0 & \lambda^2 & 3\lambda^2 & \lambda^2 \\ \lambda^2 & -\lambda^2 & 2\lambda^2 & \lambda^2 \end{pmatrix}.$$

1002.
$$A=\begin{pmatrix} 3\lambda^3-6\lambda^2+\lambda+3 & 2\lambda^3-4\lambda^2+3\lambda-1 & \lambda^3-2\lambda^2+\lambda \\ 3\lambda^2-8\lambda+5 & 2\lambda^2-4\lambda+1 & \lambda^2-2\lambda+1 \\ 3\lambda^3-3\lambda^2-5\lambda+6 & 2\lambda^3-2\lambda^2-\lambda+1 & \lambda^3-\lambda^2-\lambda+1 \end{pmatrix};$$

$$B=\begin{pmatrix} 3\lambda^3-9\lambda^2+7\lambda+1 & 2\lambda^3-6\lambda^2+7\lambda-2 \\ 3\lambda^3-9\lambda^2+9\lambda-5 & 2\lambda^3-6\lambda^2+6\lambda-1 \\ 3\lambda^3-9\lambda^2+5\lambda+5 & 2\lambda^3-6\lambda^2+8\lambda-2 \\ \lambda^3-3\lambda^2+3\lambda-1 & \\ \lambda^3-3\lambda^2+3\lambda-1 & \\ \lambda^3-3\lambda^2+3\lambda-1 & \end{pmatrix};$$

$$C=\begin{pmatrix} 3\lambda^2-3\lambda+1 & \lambda^2-\lambda & 0 \\ 2\lambda^2-\lambda-1 & 7\lambda-3\lambda^2-4 & \lambda^2-2\lambda+1 \\ 5\lambda^2-7\lambda+3 & 5\lambda-2\lambda^2-3 & \lambda^2-2\lambda+1 \end{pmatrix}.$$

1003. A λ-matrix is said to be *unimodular* if its determinant is a polynomial of zero degree in λ, that is to say, a nonzero constant. Find the normal diagonal form of a unimodular λ-matrix.

1004. Prove that the inverse of a λ-matrix is a λ-matrix if and only if the given matrix A is unimodular.

***1005.** Prove the following statement: for two rectangular λ-matrices A and B, each consisting of m rows and n columns, to be equivalent, it is necessary and sufficient that the following equation hold true: $B = PAQ$, where P and Q are unimodular λ-matrices of order m and n respectively. Show that the required matrices P and Q can be found thus: after finding a series of elementary transformations that carry A into B, apply all the transformations of the rows in the same order to the unit matrix E_m of order m and all the transformations of the columns in the same order to the unit matrix E_n of order n.

Use the method given in problem 1005 to find, relative to the given λ-matrix A, the unimodular matrices P and Q such that the matrix $B = PAQ$ has a normal diagonal form (the matrices P and Q are not defined uniquely):

1006. $$A=\begin{pmatrix} \lambda^2-\lambda+4 & \lambda^2+3 \\ \lambda^2-2\lambda+3 & \lambda^2-\lambda+2 \end{pmatrix}.$$

1007. $$A=\begin{pmatrix} \lambda^4+4\lambda^3+4\lambda^2+\lambda+2 & \lambda^3+4\lambda^2+4\lambda \\ \lambda^4+5\lambda^3+8\lambda^2+5\lambda+2 & \lambda^3+5\lambda^2+8\lambda+4 \end{pmatrix}.$$

1008.

$$A=\begin{pmatrix} \lambda^4+3\lambda^3-5\lambda^2+\lambda+1 & 2\lambda^4+3\lambda^3-5\lambda^2+\lambda-1 \\ \lambda^4-\lambda^3+1 & 2\lambda^4-\lambda^3-\lambda^2 \\ \lambda^4+2\lambda^3-4\lambda^2+\lambda+1 & 2\lambda^4+2\lambda^3-4\lambda^2+\lambda-1 \\ 2\lambda^4+2\lambda^3-4\lambda^2 & \\ 2\lambda^4-2\lambda^3 & \\ 2\lambda^4+\lambda^3-3\lambda^2 & \end{pmatrix}.$$

For given λ-matrices A and B find the unimodular λ-matrices P and Q that satisfy the equation $B = PAQ$ (the matrices P and Q are not defined uniquely—see problem 1005):

1009.

$$A=\begin{pmatrix} 2\lambda^2-\lambda+1 & 3\lambda^2-2\lambda+1 \\ 2\lambda^2+\lambda-1 & 3\lambda^2+\lambda-2 \end{pmatrix};$$

$$B=\begin{pmatrix} 3\lambda^3+7\lambda+2 & 3\lambda^2+4\lambda-1 \\ 2\lambda^3+5\lambda+1 & 2\lambda^3+3\lambda-1 \end{pmatrix}.$$

1010.

$$A=\begin{pmatrix} 2\lambda^2-\lambda-1 & 2\lambda^3+\lambda^2-3\lambda \\ \lambda^2-\lambda & \lambda^3-\lambda \end{pmatrix};$$

$$B=\begin{pmatrix} \lambda^3-\lambda^2+\lambda-1 & 2\lambda^3-\lambda^2+\lambda-2 \\ \lambda^3-\lambda^2 & 2\lambda^3-\lambda^2-\lambda \end{pmatrix}.$$

1011.

$$A=\begin{pmatrix} \lambda^2+\lambda-1 & \lambda+1 & \lambda^2-2 \\ \lambda^3+2\lambda^2 & \lambda^2+2\lambda+1 & \lambda^3+\lambda^2-2\lambda-1 \\ \lambda^3+\lambda^2-\lambda+1 & \lambda^2+\lambda & \lambda^3-2\lambda+1 \end{pmatrix};$$

$$B=\begin{pmatrix} 4\lambda+3 & 2\lambda+2 & 2\lambda^2-2\lambda-3 \\ 10\lambda+2 & 5\lambda+5 & 5\lambda^2-5\lambda-2 \\ 4\lambda^2-7\lambda-8 & 2\lambda^2-3\lambda-5 & 2\lambda^3-7\lambda^2+2\lambda+8 \end{pmatrix}.$$

1012.

$$A=\begin{pmatrix} \lambda^3-\lambda^2-\lambda+1 & 2\lambda^2+2\lambda & \lambda^3+\lambda^2 \\ 2\lambda^3-3\lambda^2-3\lambda+2 & 5\lambda^2+5\lambda & 2\lambda^3+2\lambda^2 \\ \lambda^3-\lambda & \lambda^2+\lambda & \lambda^3+\lambda^2 \end{pmatrix};$$

$$B=\begin{pmatrix} \lambda^2+2\lambda+1 & \lambda^2+\lambda & 0 \\ 2\lambda^2+3\lambda+1 & \lambda^3+3\lambda^2+2\lambda & \lambda^3+\lambda^2 \\ 3\lambda^2+5\lambda+2 & 2\lambda^3+5\lambda^2+3\lambda & 2\lambda^3+2\lambda^2 \end{pmatrix}.$$

1013. $A=\begin{pmatrix} \lambda^2+3\lambda-4 & \lambda^2+2\lambda-3 & \lambda^2+\lambda-2 \\ 2\lambda^2+3\lambda-5 & 2\lambda^2+2\lambda-4 & 2\lambda^2+\lambda-3 \end{pmatrix}$;

$$B=\begin{pmatrix} 2\lambda^2+2\lambda-4 & 3\lambda^2+2\lambda-5 & \lambda^2+2\lambda-3 \\ 2\lambda^2+\lambda-3 & 3\lambda^2+\lambda-4 & \lambda^2+\lambda-2 \end{pmatrix}.$$

1014. $$A=\begin{pmatrix} \lambda^3+6\lambda^2+6\lambda+5 & \lambda^3+4\lambda^2+4\lambda+3 \\ \lambda^3+3\lambda^2+3\lambda+2 & \lambda^3+2\lambda^2+2\lambda+1 \\ 2\lambda^3+3\lambda^2+3\lambda+1 & 2\lambda^3+2\lambda^2+2\lambda \end{pmatrix};$$

$$B=\begin{pmatrix} \lambda^3+\lambda^2+\lambda & 2\lambda^3+\lambda^2+\lambda-1 \\ 3\lambda^3+2\lambda^2+2\lambda-1 & 6\lambda^3+2\lambda^2+2\lambda-4 \\ \lambda^3-\lambda^2-\lambda-2 & 2\lambda^3-\lambda^2-\lambda-3 \end{pmatrix}.$$

Find the invariant factors of the following λ-matrices:

1015. $$\begin{pmatrix} 3\lambda^2+2\lambda-3 & 2\lambda-1 & \lambda^2+2\lambda-3 \\ 4\lambda^2+3\lambda-5 & 3\lambda-2 & \lambda^2+3\lambda-4 \\ \lambda^2+\lambda-4 & \lambda-2 & \lambda-1 \end{pmatrix}.$$

1016. $$\begin{pmatrix} 3\lambda^3-2\lambda+1 & 2\lambda^2+\lambda-1 & 3\lambda^3+2\lambda^2-2\lambda-1 \\ 2\lambda^3-2\lambda & \lambda^2-1 & 2\lambda^3+\lambda^2-2\lambda-1 \\ 5\lambda^3-4\lambda+1 & 3\lambda^2+\lambda-2 & 5\lambda^3+3\lambda^2-4\lambda-2 \end{pmatrix}.$$

1017. $$\begin{Bmatrix} 2\lambda^3-\lambda^2+2\lambda-1 & 2\lambda^3-3\lambda^2+2\lambda-3 \\ \lambda^3+\lambda^2+\lambda+1 & \lambda^3-3\lambda^2+\lambda-3 \\ \lambda^3-2\lambda^2+\lambda-2 & \lambda^3+\lambda \\ 3\lambda^3-2\lambda^2+3\lambda-2 & 3\lambda^3-4\lambda^2+3\lambda-4 \\ \lambda^3-2\lambda^2+\lambda-2 & 5\lambda^3-2\lambda^2+5\lambda-2 \\ -\lambda^3-\lambda^2-\lambda-1 & 7\lambda^3-\lambda^2+7\lambda-1 \\ 2\lambda^3-\lambda^2+2\lambda-1 & -2\lambda^3-\lambda^2-2\lambda-1 \\ 2\lambda^3-3\lambda^2+2\lambda-3 & 6\lambda^3-3\lambda^2+6\lambda-3 \end{Bmatrix}.$$

1018.
$$\begin{pmatrix} \lambda^3+\lambda^2-\lambda+3 & \lambda^3-\lambda^2+\lambda \\ \lambda^3+3\lambda^2-3\lambda+6 & \lambda^3-3\lambda^2+3\lambda-2 \\ \lambda^3+2\lambda^2-2\lambda+4 & \lambda^3-2\lambda^2+2\lambda-1 \\ 2\lambda^3+\lambda^2-\lambda+5 & 2\lambda^3-\lambda^2+\lambda+1 \\ 2\lambda^3+\lambda^2-\lambda+4 & \lambda^3+\lambda^2-\lambda+2 \\ 2\lambda^3+3\lambda^2-3\lambda+7 & \lambda^3+3\lambda^2-3\lambda+4 \\ 2\lambda^3+2\lambda^2-2\lambda+5 & \lambda^3+2\lambda^2-2\lambda+3 \\ 4\lambda^3+\lambda^2-\lambda+7 & 2\lambda^3+\lambda^2-\lambda+3 \end{pmatrix}.$$

1019.
$$\begin{pmatrix} \lambda & 1 & 2 & 3 & \dots & n \\ 0 & \lambda & 1 & 2 & \dots & n-1 \\ 0 & 0 & \lambda & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}.$$

***1020.**
$$\begin{pmatrix} \lambda-\alpha & \beta & \beta & \beta & \dots & \beta \\ 0 & \lambda-\alpha & \beta & \beta & \dots & \beta \\ 0 & 0 & \lambda-\alpha & \beta & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda-\alpha \end{pmatrix}.$$

(the order of the matrix is n).

Elementary divisors of a λ-matrix A are the polynomials $e_1(\lambda)$, $e_2(\lambda)$, . . ., $e_s(\lambda)$, with leading coefficients equal to unity, that coincide with the highest degrees of irreducible factors, which degrees enter into the expansions of the invariant factors $E_1(\lambda)$, $E_2(\lambda)$, . . ., $E_n(\lambda)$ of matrix A into irreducible factors. Here, the collection of elementary divisors of A contains each polynomial $E_i(\lambda)$ as many times as there are invariant factors $E_k(\lambda)$ in the factorization of the polynomial. The expansion into irreducible factors is taken over the field over which we consider polynomials that are elements of the matrix A. From now on, unless otherwise stated, we consider elementary divisors over the field of complex numbers, that is, the highest degrees of polynomials of the form $\lambda - \alpha$ that enter into the expansions of invariant factors of the matrix A into linear factors.

Find the elementary divisors of the following λ-matrices:

1021. $\begin{pmatrix} \lambda^3+2 & \lambda^3+1 \\ 2\lambda^3-\lambda^2-\lambda+3 & 2\lambda^3-\lambda^2-\lambda+2 \end{pmatrix}$.

1022. $\begin{pmatrix} \lambda^3-2\lambda^2+2\lambda-1 & \lambda^2-2\lambda+1 \\ 2\lambda^3-2\lambda^2+\lambda-1 & 2\lambda^2-2\lambda \end{pmatrix}$.

1023. $\begin{pmatrix} \lambda^2+2 & 2\lambda+1 & \lambda^2+1 \\ \lambda^2+4\lambda+4 & 2\lambda+3 & \lambda^2+4\lambda+3 \\ \lambda^2-4\lambda+3 & 2\lambda-1 & \lambda^2-4\lambda+2 \end{pmatrix}$.

1024. $\begin{Vmatrix} \lambda^2-2\lambda-8 & \lambda^2+4\lambda+4 \\ \lambda^4+\lambda^3-\lambda-10 & 2\lambda^2+5\lambda+2 \\ \lambda^4+\lambda^3-2\lambda^2-3\lambda-6 & \lambda^2+4\lambda+4 \\ \lambda^4+\lambda^3-\lambda^2+\lambda-2 & \lambda^2+\lambda-2 \\ \lambda^2-4 & \lambda^2-3\lambda-10 \\ \lambda^3+3\lambda^2-\lambda-6 & \lambda^4+\lambda^3-2\lambda-12 \\ \lambda^4+\lambda^3-2\lambda^2-\lambda-2 & \lambda^4+\lambda^3-2\lambda^2-4\lambda-8 \\ \lambda^3+2\lambda^2-\lambda-2 & \lambda^4+\lambda^3-\lambda^2+\lambda-2 \end{Vmatrix}$.

1025. $\begin{pmatrix} \lambda^2-2 & \lambda^2+\lambda+3 & \lambda^2+2 & \lambda^2-3 \\ \lambda^2+3\lambda-1 & \lambda^2+3\lambda+3 & \lambda^2+2\lambda+1 & \lambda^2+3\lambda-2 \\ 2\lambda^2-4 & \lambda^2+\lambda+4 & \lambda^2+3 & 2\lambda^2-5 \\ 2\lambda^2+3\lambda-3 & \lambda^2+3\lambda+4 & \lambda^2+2\lambda+2 & 2\lambda^2+3\lambda-4 \end{pmatrix}$.

Find the elementary divisors of the following λ-matrices in the fields of rational, real, and complex numbers:

1026. $\begin{pmatrix} \lambda^2+2 & \lambda^2+1 & 2\lambda^2-2 \\ \lambda^2+1 & \lambda^2+1 & 2\lambda^2-2 \\ \lambda^2+2 & \lambda^2+1 & 3\lambda^2-5 \end{pmatrix}$.

1027. $\begin{pmatrix} 2\lambda^2+3 & \lambda^2+1 & \lambda^6+6\lambda^4+\lambda^2+2 \\ 4\lambda^2+11 & 2\lambda^2+5 & 2\lambda^6+12\lambda^4+2\lambda^2-26 \\ 2\lambda^2+3 & \lambda^2+1 & 2\lambda^6+12\lambda^4+\lambda^2-30 \end{pmatrix}$.

1028.

$$\begin{pmatrix} \lambda^4+1 & \lambda^7-\lambda^4+\lambda^3-1 & \lambda^4-4\lambda^3+4\lambda-5 \\ 2\lambda^4+3 & 2\lambda^7-2\lambda^4+4\lambda^3-2 & 3\lambda^4-10\lambda^3+\lambda^2+10\lambda-14 \\ \lambda^4+2 & \lambda^7-\lambda^4+2\lambda^3-2 & 2\lambda^4-6\lambda^3+\lambda^2+6\lambda-9 \end{pmatrix}.$$

Find the normal diagonal form of a square λ-matrix if we know the elementary divisors, the rank r, and the order n:

1029. $\lambda+1$, $\lambda+1$, $(\lambda+1)^2$, $\lambda-1$, $(\lambda-1)^2$; $r=4$, $n=5$.

1030. $\lambda+2$, $(\lambda+2)^2$, $(\lambda+2)^3$, $\lambda-2$, $(\lambda-2)^2$; $r=n=4$.

1031. $\lambda-1$, $\lambda-1$, $(\lambda-1)^3$, $\lambda+2$, $(\lambda+2)^2$; $r=4$, $n=5$.

*1032. Prove that the set of elementary divisors of a diagonal λ-matrix is obtained via the union (with appropriate repetitions) of the sets of elementary divisors of all diagonal elements of that matrix.

*1033. Prove that the set of elementary divisors of a block diagonal λ-matrix is equal to the union (with appropriate repetitions) of the sets of the elementary divisors of all its diagonal blocks.

Using problem 1032 or 1033, find the normal diagonal form of the following λ-matrices:

1034.

$$\begin{pmatrix} \lambda(\lambda-1)^2 & 0 & 0 & 0 \\ 0 & \lambda^2(\lambda+1) & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 \\ 0 & 0 & 0 & \lambda(\lambda+1)^3 \end{pmatrix}.$$

1035.

$$\begin{pmatrix} \lambda^2-4 & 0 & 0 & 0 \\ 0 & \lambda^2+2\lambda & 0 & 0 \\ 0 & 0 & \lambda^3-2\lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3-4\lambda \end{pmatrix}.$$

1036.

$$\begin{pmatrix} 0 & 0 & 0 & \lambda^3+6\lambda^2+9\lambda \\ 0 & 0 & \lambda^3+\lambda^2-6\lambda & 0 \\ 0 & \lambda^2-4\lambda+4 & 0 & 0 \\ \lambda^4+\lambda^3-6\lambda^2 & 0 & 0 & 0 \end{pmatrix}.$$

1037.

$$\begin{pmatrix} 0 & 0 & 0 & \lambda^4+2\lambda^2-2\lambda-1 \\ 0 & 0 & \lambda^4-2\lambda^2+2\lambda-1 & 0 \\ 0 & \lambda^2+2\lambda+1 & 0 & 0 \\ \lambda^2-2\lambda+1 & 0 & 0 & 0 \end{pmatrix}.$$

1038.

$$\begin{pmatrix} \lambda^2+2\lambda-3 & \lambda^2+\lambda-2 & 0 & 0 \\ 2\lambda^2+2\lambda-4 & 2\lambda^2+\lambda-3 & 0 & 0 \\ 0 & 0 & \lambda+1 & \lambda+2 \\ 0 & 0 & \lambda^2-1 & \lambda^2+\lambda-2 \end{pmatrix}.$$

1039.

$$\begin{pmatrix} \lambda^2-\lambda-2 & \lambda^3+\lambda^2-\lambda-1 & 0 & 0 \\ \lambda^2-4 & \lambda^3+2\lambda^2-\lambda-2 & 0 & 0 \\ 0 & 0 & \lambda^2+2\lambda & \lambda^2+6\lambda-2 \\ 0 & 0 & \lambda^2+\lambda-2 & \lambda^2+5\lambda-7 \end{pmatrix}.$$

1040.

$$\begin{pmatrix} 0 & 0 & \lambda^3-\lambda^2-\lambda-2 & \lambda^3-2\lambda-4 \\ 0 & 0 & \lambda^3+\lambda^2-6\lambda & \lambda^2+\lambda-6 \\ \lambda^2-2\lambda+1 & \lambda-2 & 0 & 0 \\ \lambda^3-2\lambda^2+6\lambda-1 & \lambda^2-2\lambda+5 & 0 & 0 \end{pmatrix}.$$

1041.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda^2-2\lambda+1 \\ \lambda^2+2\lambda-3 & \lambda^2+\lambda-2 & 0 \\ \lambda^3+2\lambda^2+\lambda-4 & \lambda^3+2\lambda^2-3 & 0 \\ \lambda^2-2\lambda-3 & \lambda^3+\lambda^2-9\lambda-9 & \\ \lambda^2-\lambda-2 & \lambda^3+2\lambda^2-5\lambda-6 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{pmatrix}.$$

Defining the equivalence and normal diagonal form of integral matrices as is done in problems 942 and 943, find the greatest common divisors D_k of the kth-order minors of the following matrices by means of reducing them to normal diagonal form via elementary operations:

1042. $\begin{pmatrix} 0 & 2 & 4 & -1 \\ 6 & 12 & 14 & 5 \\ 0 & 4 & 14 & -1 \\ 10 & 6 & -4 & 11 \end{pmatrix}$.

1043. $\begin{pmatrix} 0 & 6 & -9 & -3 \\ 12 & 24 & 9 & 9 \\ 30 & 42 & 45 & 27 \\ 66 & 78 & 81 & 63 \end{pmatrix}$.

1044. Prove that by means of elementary transformations of rows alone (or columns alone) it is possible to reduce any λ-matrix of rank r to triangular or trapezoidal form with the zeros obtained either above or below the main diagonal, at one's pleasure; the nonzero elements will lie only in the first r rows (or the first r columns).

***1045.** Prove that every nonsingular λ-matrix A may be expressed in the form $A = PR$, where P is a unimodular λ-matrix and R is a triangular λ-matrix whose elements on the main diagonal have leading coefficient unity, those below the main diagonal are zero, and those above the main diagonal are of a power less than that of the element of the main diagonal of the same column (or are zero); this representation is unique.

Sec. 14. Similar Matrices. Characteristic and Minimal Polynomials. Jordan and Diagonal Forms of a Matrix. Functions of Matrices

All problems in this section are posed in matrix form. In particular, the properties of characteristic roots (eigenvalues) of a matrix and the reduction of the matrix to the Jordan form are considered without regard for the properties of the eigenvectors and the invariant subspaces of the appropriate linear transformation. This relationship (in particular, finding the basis in which the matrix of the

given linear transformation has the Jordan form) is considered in Chapter IV. This does not prevent using the problems of this section when studying the properties of linear transformations to the extent in which the relationship between linear transformations and their matrices in some given basis has been studied.

1046. A matrix A is said to be *similar* to a matrix B (this is denoted as $A \sim B$) if there exists a nonsingular matrix T such that $B = T^{-1}AT$. Show that the similarity relation has the following properties:

(a) $A \sim A$; (b) if $A \sim B$, then $B \sim A$;

(c) if $A \sim B$ and $B \sim C$, then $A \sim C$.

1047. Prove that if at least one of the two matrices A and B is nonsingular, then the matrices AB and BA are similar.

Give an example of two singular matrices A, B for which the matrices AB and BA are not similar.

***1048.** Find all matrices each of which is similar only to itself.

1049. Let a matrix B be obtained from A by interchanging the ith and jth rows and also the ith and jth columns. Prove that A and B are similar and find the nonsingular matrix T for which $B = T^{-1}AT$.

***1050.** Show that the matrix A is similar to the matrix B obtained from A by a reflection about its centre.

1051. Let $i_1, i_2, \ldots, i_n$ be any permutation of the numbers $1, 2, \ldots, n$. Prove that the matrices

$$A = \begin{pmatrix} a_{11}a_{12} & \ldots & a_{1n} \\ a_{21}a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1}a_{n2} & \ldots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_{i_1i_1}a_{i_1i_2} & \ldots & a_{i_1i_n} \\ a_{i_2i_1}a_{i_2i_2} & \ldots & a_{i_2i_n} \\ \ldots & \ldots & \ldots \\ a_{i_ni_1}a_{i_ni_2} & \ldots & a_{i_ni_n} \end{pmatrix}$$

are similar.

1052. Suppose we have two similar matrices A and B. Show that the collection of all nonsingular matrices T for which $B = T^{-1}AT$ can be obtained from the collection of all nonsingular matrices, permutable with A, by postmultiplying these matrices by any one matrix T_0 with the property $B = T_0^{-1}AT_0$.

1053. Prove that if a matrix A is similar to a diagonal matrix, then the pth associated matrix A_p (see problem 969) is also similar to the diagonal matrix.

1054. Prove that if two matrices A and B are similar to diagonal matrices, then their Kronecker product $A \times B$ (see problem 963) is also a matrix similar to the diagonal matrix.

1055. Prove that if the matrices A and B are similar, then the pth associated matrices A_p and B_p (under any two arrangements of the combinations of n numbers of rows and columns taken p at a time) are also similar.

1056. Prove that if the matrices A_1, B_1 are similar, respectively, to the matrices A_2, B_2, then the Kronecker products $A_1 \times B_1$ and $A_2 \times B_2$ (under any two arrangements of pairs of indices) are also similar.

1057. Prove that if the square λ-matrix B is given in the form $B = B_0\lambda^s + B_1\lambda^{s-1} + \ldots + B_s$, where B_0, B_1, $\ldots$, B_s are matrices not dependent on λ and the matrix B_0 is nonsingular, then any square λ-matrix A of the same order as B may be divided by B on the left or on the right, that is, there exist a right quotient Q_1 and a right remainder R_1 such that $A = BQ_1 + R_1$ and a left quotient Q_2 and a left remainder R_2 such that $A = Q_2B + R_2$; note that the powers of the elements of the matrices R_1 and R_2 in λ are lower than s and both pairs Q_1, R_1 and Q_2, R_2 are defined uniquely.

1058. Take the matrix

$$A = \begin{pmatrix} -2\lambda^2+5\lambda+3 & -\lambda^2+3\lambda+2 & -\lambda+6 \\ -3\lambda^2+7\lambda+11 & -3\lambda^2+9\lambda+1 & -2\lambda+8 \\ -\lambda^2+2\lambda+8 & -2\lambda^2+5\lambda+3 & -\lambda+4 \end{pmatrix}$$

and divide it on the left by $B-\lambda E$, where $B = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$.

1059. Take the matrix

$$A = \begin{pmatrix} -\lambda^3+\lambda^2+3\lambda+6 & \lambda^2+2\lambda & \lambda^2+2\lambda+6 \\ -2\lambda^3+2\lambda^2+9\lambda+8 & \lambda^2+6\lambda+1 & 2\lambda^2+7\lambda+8 \\ -\lambda^3+\lambda^2+3\lambda+5 & \lambda^2+2\lambda-9 & \lambda^2+5\lambda-2 \end{pmatrix}$$

and divide it on the right by $B-\lambda E$, where $B=$

$$=\begin{pmatrix}1 & 2 & 1\\ 3 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}.$$

*1060. Prove that if two matrices A and B with numerical elements (or elements taken from any other field P) are similar, then their characteristic matrices $A-\lambda E$ and $B-\lambda E$ are equivalent.

*1061. Prove that if the characteristic matrices $A-\lambda E$ and $B-\lambda E$ of two matrices A and B are equivalent, then the matrices themselves are similar. Also show that if $B-\lambda E=P(A-\lambda E)Q$, where P and Q are unimodular λ-matrices and P_0 and Q_0 are the remainders resulting from the division of P on the left and of Q on the right by $B-\lambda E$, then $B=P_0AQ_0$ and $P_0Q_0=E$, that is, the matrix Q_0 accomplishes a similarity transformation of the matrix A into the matrix B.

1062. Prove that any square matrix A is similar to its transpose A'.

Determine whether the following matrices are similar or not:

1063.
$$A=\begin{pmatrix}3 & 2 & -5\\ 2 & 6 & -10\\ 1 & 2 & -3\end{pmatrix};\qquad B=\begin{pmatrix}6 & 20 & -34\\ 6 & 32 & -51\\ 4 & 20 & -32\end{pmatrix}.$$

1064.
$$A=\begin{pmatrix}6 & 6 & -15\\ 1 & 5 & -5\\ 1 & 2 & -2\end{pmatrix};\qquad B=\begin{pmatrix}37 & -20 & -4\\ 34 & -17 & -4\\ 119 & -70 & -11\end{pmatrix}.$$

1065.
$$A=\begin{pmatrix}4 & 6 & -15\\ 1 & 3 & -5\\ 1 & 2 & -4\end{pmatrix};\qquad B=\begin{pmatrix}1 & -3 & 3\\ -2 & -6 & 13\\ -1 & -4 & 8\end{pmatrix};$$

$$C=\begin{pmatrix}-13 & -70 & 119\\ -4 & -19 & 34\\ -4 & -20 & 35\end{pmatrix}.$$

1066.

$$A=\begin{pmatrix} 14 & -2 & -7 & -1 \\ 20 & -2 & -11 & -2 \\ 19 & -3 & -9 & -1 \\ -6 & 1 & 3 & 1 \end{pmatrix}; \quad B=\begin{pmatrix} 4 & 10 & -19 & 4 \\ 1 & 6 & -8 & 3 \\ 1 & 4 & -6 & 2 \\ 0 & -1 & 1 & 0 \end{pmatrix};$$

$$C=\begin{pmatrix} 41 & -4 & -26 & -7 \\ 14 & -13 & -91 & -18 \\ 40 & -4 & -25 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the method given in problem 1061, find, for the given matrices A and B, a nonsingular matrix T such that $B = T^{-1}AT$ (the desired matrix T is not defined uniquely):

***1067.** $A=\begin{pmatrix} 5 & -1 \\ 9 & -1 \end{pmatrix}; \quad B=\begin{pmatrix} 38 & -81 \\ 16 & -34 \end{pmatrix}.$

***1068.** $A=\begin{pmatrix} 17 & -6 \\ 45 & -16 \end{pmatrix}; \quad B=\begin{pmatrix} 14 & -60 \\ 3 & -13 \end{pmatrix}.$

1069. $A=\begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{pmatrix}; \quad B=\begin{pmatrix} 24 & -11 & -22 \\ 20 & -8 & -20 \\ 12 & -6 & -10 \end{pmatrix}.$

***1070.** Prove that the coefficients of the characteristic polynomial $|A - \lambda E|$ of the matrix A can be expressed in terms of the elements of that matrix in the following manner:

$$|A - \lambda E| = (-\lambda)^n + c_1 (-\lambda)^{n-1} + \\ + c_2 (-\lambda)^{n-2} + \ldots + c_n,$$

where c_k is the sum of all principal minors of order k of matrix A (a minor is a principal minor if the numbers of the rows in it coincide with the numbers of the columns).

***1071.** Find the eigenvalues (the roots of the characteristic polynomial) of the matrix $A'A$, where $A = (a_1, a_2, \ldots, a_n)$ and A' is the transpose of A.

1072. Prove that the sum of the eigenvalues of matrix A is equal to its trace (that is, the sum of the elements of the

principal diagonal), and the product of these numbers is equal to the determinant $|A|$.

1073. Prove that all eigenvalues of matrix A are nonzero if and only if A is nonsingular.

***1074.** Let $p > 0$ be the multiplicity of the root λ_0 of the characteristic polynomial $|A - \lambda E|$ of matrix A of order n, let r be the rank and $d = n - r$ the nullity of the matrix $A - \lambda_0 E$. Prove the truth of the following inequalities:

$$1 \leqslant d = n - r \leqslant p.$$

1075. Give examples of nth-order matrices for which the first or second inequality of the preceding problem becomes an equality, that is, $d = 1$ or $d = p$.

***1076.** Prove that the eigenvalues of the inverse matrix A^{-1} are equal (with account taken of multiplicity) to the reciprocals of the eigenvalues of matrix A.

***1077.** Prove that the eigenvalues of matrix A^2 are equal (with account taken of multiplicity) to the squares of the eigenvalues of matrix A.

***1078.** Prove that the eigenvalues of matrix A^p are equal (with account taken of multiplicity) to the pth powers of the eigenvalues of the matrix A.

***1079.** Let $\varphi(\lambda) = |A - \lambda E|$ be the characteristic polynomial of matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A; also let $f(\lambda)$ be an arbitrary polynomial. Prove that the determinant of the matrix $f(A)$ satisfies the equality $|f(A)| = f(\lambda_1) f(\lambda_2) \ldots f(\lambda_n) = R(f, \varphi)$, where $R(f, \varphi)$ is the resultant of the polynomials f and φ. But if the characteristic polynomial is defined as $\varphi(\lambda) = |\lambda E - A|$, then

$$|f(A)| = R(\varphi, f).$$

Recall that the *resultant* of two polynomials

$$f(x) = a_0 \prod_{i=1}^{n} (x - \alpha_i) \text{ and } g(x) = b_0 \prod_{j=1}^{s} (x - \beta_j)$$

is the number

$$R(f, g) = a_0^s b_0^n \prod_{i=1}^{n} \prod_{j=1}^{s} (\alpha_i - \beta_j) = a_0^s \prod_{i=1}^{n} g(\alpha_i) =$$

$$= (-1)^{ns} b_0^n \prod_{j=1}^{s} f(\beta_j).$$

*1080. Prove that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of matrix A and $f(x)$ is a polynomial, then $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ are eigenvalues of the matrix $f(A)$.

*1081. Prove that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of matrix A and $f(x) = \frac{g(x)}{h(x)}$ is a rational function defined for the value $x = A$ (that is, the function satisfies the condition $|h(A)| \neq 0$), then $|f(A)| = f(\lambda_1) f(\lambda_2) \ldots f(\lambda_n)$ and the numbers $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ are the eigenvalues of the matrix $f(A)$.

*1082. Prove that if A and B are square matrices of the same order, then the characteristic polynomials of the matrices AB and BA coincide.

*1083. Find the eigenvalues of the cyclic matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_2 & a_3 & a_4 & \ldots & a_1 \end{pmatrix}.$$

*1084. Find the eigenvalues of the nth-order matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

*1085. A *Jordan matrix* is a block-diagonal matrix with diagonal blocks of the form

$$\begin{pmatrix} \alpha & 1 & 0 & \ldots & 0 \\ 0 & \alpha & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \alpha \end{pmatrix}.$$

These are the so-called Jordan submatrices. The Jordan form of the matrix A is a Jordan matrix A_J similar to the matrix A.

Taking advantage of the theorem which states that the collection of elementary divisors of a block-diagonal matrix is equal to the union of the collections of elementary divisors

of its diagonal blocks (see problem 1033), prove that any matrix A has, over the field of complex numbers (or over any field containing all the eigenvalues of A), a Jordan form which is unique up to the order of the blocks.

Write the Jordan form A_J of the matrix A if we have the invariant factors $E_i(\lambda)$ $(i = 1, 2, \ldots, n)$ of its characteristic matrix $A - \lambda E$:

1086. $E_1(\lambda) = E_2(\lambda) = 1$, $E_3(\lambda) = E_4(\lambda) = \lambda - 1$, $E_5(\lambda) = E_6(\lambda) = (\lambda - 1)(\lambda + 2)$.

1087. $E_1(\lambda) = E_2(\lambda) = E_3(\lambda) = 1$, $E_4(\lambda) = \lambda + 1$, $E_5(\lambda) = (\lambda + 1)^2$, $E_6(\lambda) = (\lambda + 1)^2(\lambda - 5)$.

1088. $E_1(\lambda) = E_2(\lambda) = 1$, $E_3(\lambda) = \lambda - 2$, $E_4(\lambda) = \lambda^2 - 4$.

1089. Prove that for any square λ-matrix $A(\lambda)$ of order n, the determinant of which is a polynomial in λ of degree n, there exists a numerical matrix B of order n such that the matrix $A(\lambda)$ is equivalent to the characteristic matrix $B - \lambda E$.

Find the Jordan form of each of the following matrices:

1090. $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$.

1091. $\begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}$.

1092. $\begin{pmatrix} 9 & -6 & -2 \\ 18 & -12 & -3 \\ 18 & -9 & -6 \end{pmatrix}$.

1093. $\begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}$.

1094. $\begin{pmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 1 & -2 & -2 \end{pmatrix}$.

1095. $\begin{pmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{pmatrix}$.

1096. $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$.

1097. $\begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}$.

1098. $\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$.

1099. $\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$.

1100. $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 3 & -7 & 7 \end{pmatrix}$.

1101. $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \alpha & 0 & \alpha \end{pmatrix}$, where $\alpha \neq 0$.

1102. $\begin{pmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{pmatrix}$.

1103. $\begin{pmatrix} 4 & 6 & -15 \\ 3 & 4 & -12 \\ 2 & 3 & -8 \end{pmatrix}$.

1104. $\begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}$.

1105. $\begin{pmatrix} 1 & -3 & 0 & 3 \\ -2 & -6 & 0 & 13 \\ 0 & -3 & 1 & 3 \\ -1 & -4 & 0 & 8 \end{pmatrix}$.

1106. $\begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$.

1107. $\begin{pmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

1108. $\begin{pmatrix} 3 & -1 & 1 & -7 \\ 9 & -3 & -7 & -1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 2 & -4 \end{pmatrix}$.

1109. $\begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$ (matrix of order n).

1110. $\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ (matrix of order n).

1111.
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 0 & 1 & 2 & 3 & \dots & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

1112.
$$\begin{pmatrix} n & n-1 & n-2 & \dots & 1 \\ 0 & n & n-1 & \dots & 2 \\ 0 & 0 & n & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}.$$

1113.
$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & 0 & \dots & 0 \\ 1 & 2 & 3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & 4 & \dots & n \end{pmatrix}.$$

1114.
$$\begin{pmatrix} 0 & \alpha & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha \\ \alpha & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

1115. Find the Jordan form of a matrix of the form
$$\begin{pmatrix} \alpha & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \alpha & a_{23} & \dots & a_{2n} \\ 0 & 0 & \alpha & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$$
provided that $a_{12} a_{23} \dots a_{n-1,n} \neq 0$.

1116. Prove that if the characteristic polynomial $|A - \lambda E|$ of a matrix A does not have any multiple roots, then A is similar to a diagonal matrix (the elements of the matrix T that transforms A into diagonal form belong to the field that contains all eigenvalues of A).

1117. Prove that a matrix A over a given field P is similar to a diagonal matrix if and only if the last invariant factor $E_n(\lambda)$ of the characteristic matrix $A - \lambda E$ does not have any multiple roots and all its roots belong to the field P.

Determine whether the following matrices are similar to diagonal matrices in the fields of rational, real, and complex

numbers:

1118. $\begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix}$. **1119.** $\begin{pmatrix} 8 & 15 & -36 \\ 8 & 21 & -46 \\ 5 & 12 & -27 \end{pmatrix}$.

1120. $\begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}$. **1121.** $\begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}$.

1122. Prove that if the last invariant factor $E_n(\lambda)$ of the characteristic matrix $A - \lambda E$ of a matrix A of order n is of the power n, then all diagonal elements of distinct blocks of the Jordan form of the matrix A are distinct.

1123. Prove that a matrix A is nilpotent (i.e. $A^k = 0$ for any positive integer k) if and only if all its eigenvalues are zero.

1124. Prove that a nonzero nilpotent matrix cannot be brought to diagonal form by a similarity transformation.

1125. Find the Jordan form of the idempotent matrix A (a matrix is idempotent if it has the property $A^2 = A$).

1126. Prove that an involutory matrix A (that is, a matrix having the property $A^2 = E$) is similar to a diagonal matrix and find the form of this diagonal matrix.

1127. Prove that a periodic matrix A (that is, a matrix with the property that $A^k = E$ for any positive integer k) is similar to a diagonal matrix and find the form of that diagonal matrix.

1128. Find the minimal polynomial (the definition of which is given in problem 830): (a) of a unit matrix, (b) of a zero matrix.

1129. For what matrices does the minimal polynomial have the form $\lambda - \alpha$, where α is a scalar?

1130. Find the minimal polynomial of a Jordan submatrix of order n with the number α on the diagonal.

1131. Prove that the minimal polynomial of a block-diagonal matrix is equal to the lowest common multiple of the minimal polynomials of its diagonal blocks.

1132. Prove that the minimal polynomial of a matrix A is equal to the last invariant factor $E_n(\lambda)$ of its characteristic matrix $A - \lambda E$.

1133. Prove that some degree of the minimal polynomial of the matrix A is divisible by the characteristic polynomial of that matrix.

Find the minimal polynomials of the following matrices:

1134. $\begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. **1135.** $\begin{pmatrix} 4 & -2 & 2 \\ -5 & 7 & -5 \\ -6 & 7 & -4 \end{pmatrix}$.

1136. Prove that for two matrices to be similar it is necessary (but not sufficient) that they have the same characteristic and minimal polynomials. Give an example of two nonsimilar matrices which have the same characteristic polynomial $\varphi(\lambda)$ and the same minimal polynomial $\psi(\lambda)$.

1137. Find the kth power A^k of the nth-order Jordan submatrix

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha \end{pmatrix}.$$

***1138.** Prove that the value of the polynomial $f(x)$ in the Jordan submatrix A of order n with the number α on the main diagonal

$$A = \begin{Bmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha \end{Bmatrix}$$

is given by the formula

$$f(A) = \begin{pmatrix} f(\alpha) & \frac{f'(\alpha)}{1!} & \frac{f''(\alpha)}{2!} & \frac{f'''(\alpha)}{3!} & \dots & \frac{f^{(n-1)}(\alpha)}{(n-1)!} \\ 0 & f(\alpha) & \frac{f'(\alpha)}{1!} & \frac{f''(\alpha)}{2!} & \dots & \frac{f^{(n-2)}(\alpha)}{(n-2)!} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & f(\alpha) \end{pmatrix}.$$

1139. Solve problem 1080 by using the Jordan form of the matrix A.

1140. Find the Jordan form of the square of a Jordan submatrix, the diagonal of which has the number $\alpha \neq 0$.

***1141.** Find the Jordan form of the square of a Jordan submatrix with zero on the main diagonal (a nilpotent Jordan submatrix).

1142. Let X_J be the Jordan form of the matrix X. Prove that $(A + \alpha E)_J = A_J + \alpha E$, where A is any square matrix and α is a scalar.

***1143.** Find the Jordan form of the matrix

$$\begin{pmatrix} \alpha & 0 & 1 & 0 & \dots & 0 \\ 0 & \alpha & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha \end{pmatrix}$$

of order $n \geqslant 3$.

***1144.** Prove that any square matrix can be represented in the form of a product of two symmetric matrices, one of which is nonsingular.

***1145.** Knowing the eigenvalues of the matrix A, find the eigenvalues of the pth associated matrix A_p (the definition of which is given in problem 969).

***1146.** Knowing the eigenvalues of two square matrices, A of order p and B of order q, find the eigenvalues of their Kronecker product $A \times B$ (the definition of which is given in problem 963).

***1147.** Let $\psi(\lambda) = (\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \dots (\lambda - \lambda_s)^{r_s}$ be the minimal polynomial of a matrix A of degree $r = r_1 + r_2 + \dots + r_s$. Here, r_k is the multiplicity of λ_k as a root of the minimal polynomial $\psi(\lambda)$.

If the function $f(\lambda)$ has the numbers

$$f(\lambda_k),\ f'(\lambda_k),\ f''(\lambda_k),\ \dots,\ f^{(r_k-1)}(\lambda_k) \qquad (k = 1, 2, \dots, s), \tag{1}$$

then we say that the function $f(\lambda)$ *is defined on the spectrum of the matrix* A and the set of numbers (1) is called *the set of values of the function* $f(\lambda)$ *on the spectrum of the matrix* A. Prove that the values of the polynomials $g(\lambda)$ and $h(\lambda)$ of the matrix A coincide, that is, $g(A) = h(A)$ if and only if the values of these polynomials coincide on the spectrum of A.

1148. Suppose a function $f(\lambda)$ is defined on the spectrum of a matrix A (in the sense of the preceding problem). Prove that if there is at least one polynomial whose value on the spectrum of A coincides with the values of $f(\lambda)$, then there is an infinity of such polynomials and there is only one of them that has a degree less than the degree of the minimal polynomial of matrix A. This polynomial $r(\lambda)$ is termed the *Lagrange-Sylvester interpolation polynomial* of the function $f(\lambda)$ on the spectrum of the matrix A. Its value of the matrix A is assumed, by definition, to be the value of the function $f(\lambda)$ of that matrix: $f(A) = r(A)$.

1149. Prove that if the function $f(\lambda)$ is defined on the spectrum of a matrix A and the characteristic polynomial $|A - \lambda E|$ does not have any multiple roots, then the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ exists and, hence, the matrix $f(A)$ is meaningful. Find the aspects of $r(\lambda)$ and $f(A)$.

1150. Prove that if the function $f(\lambda)$ is defined on the spectrum of a matrix A and the minimal polynomial of that matrix $\psi(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_s)$ has no multiple roots, then the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ exists and the matrix $f(A)$ is meaningful. Find the expression for computing $f(A)$.

***1151.** Prove that if the function $f(\lambda)$ is defined on the spectrum of a matrix A, then the definition of the matrix $f(A)$ (which was given in problem 1148) is meaningful, that is, there is a Lagrange-Sylvester interpolation polynomial $r(\lambda)$. Let $\psi(\lambda) = (\lambda - \lambda_1)^{r_1} \ldots (\lambda - \lambda_s)^{r_s}$ be the minimal polynomial of the matrix A, where the roots $\lambda_1, \ldots, \lambda_s$ are distinct and

$$\psi_k(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_k)^{r_k}} \qquad (k = 1, 2, \ldots, s).$$

Show that

$$r(\lambda) = \sum_{k=1}^{s} [\alpha_{k,1} + \alpha_{k,2}(\lambda - \lambda_k) + \ldots + \alpha_{k,r_k}(\lambda - \lambda_k)^{r_k - 1}] \cdot \psi_k(\lambda), \quad (1)$$

where the numbers $\alpha_{k,j}$ are defined by the equations

$$\alpha_{k,j} = \frac{1}{(j-1)!} \left[\frac{f(\lambda)}{\psi_k(\lambda)}\right]^{(j-1)}_{\lambda = \lambda_k} \qquad (j = 1, 2, \ldots, r_k;\ k = 1, 2, \ldots, s). \quad (2)$$

That is to say, the expression in square brackets in (1) is equal to the sum of the first r_k terms of the Taylor expansion in powers of the difference $\lambda - \lambda_k$ for the function $\frac{f(\lambda)}{\psi_k(\lambda)}$.

1152. Let $\psi(\lambda) = (\lambda - \lambda_1)^2 (\lambda - \lambda_2)^3$ $(\lambda_1 \neq \lambda_2)$ be the minimal polynomial of a matrix A and let $f(\lambda)$ be a function defined on the spectrum of that matrix. Write the expression for the matrix $f(A)$ using the preceding problem.

1153. Prove that if the matrices A and B are similar, and $B = T^{-1}AT$ and also for the function $f(\lambda)$ the matrix $f(A)$ exists, then the matrix $f(B)$ also exists and is similar to $f(A)$, and we also have $f(B) = T^{-1}f(A)\,T$ with the same matrix T.

***1154.** Prove that if a matrix A is a block-diagonal matrix

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_s \end{pmatrix}$$

and the function $f(\lambda)$ is defined on the spectrum of A, then

$$f(A) = \begin{pmatrix} f(A_1) & & & 0 \\ & f(A_2) & & \\ & & \ddots & \\ 0 & & & f(A_s) \end{pmatrix}.$$

1155. Find the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ and the value $f(A)$ of the function $f(\lambda)$ for the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

For what functions $f(\lambda)$ is the value of $f(A)$ meaningful?

1156. Solve the same problem as in 1155 for the matrix

$$A = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha & 1 \\ 0 & 0 & 0 & \dots & 0 & \alpha \end{pmatrix}.$$

1157. Show that if a matrix A is similar to the diagonal matrix

$$A = T^{-1}\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} T$$

and for the function $f(\lambda)$ the matrix $f(A)$ exists, then $f(A)$ too is similar to the diagonal matrix, and we have

$$f(A) = T^{-1}\begin{pmatrix} f(\lambda_1) & & 0 \\ & f(\lambda_2) & \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} T$$

with the same matrix T.

1158. Prove that if $f(\lambda) = g(\lambda) + h(\lambda)$ and the matrices $g(A)$ and $h(A)$ exist, then the matrix $f(A)$ also exists, and we have $f(A) = g(A) + h(A)$.

1159. Prove that if $f(\lambda) = g(\lambda)\, h(\lambda)$ and the matrices $g(A)$ and $h(A)$ exist, then the matrix $f(A)$ exists as well, and we have $f(A) = g(A)\, h(A)$.

***1160.** Show that the function $f(\lambda) = \frac{1}{\lambda}$ is defined for all nonsingular matrices A and for them alone; note that $f(A) = A^{-1}$.

1161. Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of a matrix A and the function $f(\lambda)$ is meaningful for $\lambda = A$, then $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ will be eigenvalues of the matrix $f(A)$.

Compute the following values of the functions of the matrices, using the Lagrange-Sylvester interpolation polynomial and problems 1148 to 1152 or by finding the matrix that yields the similarity transformation of the given matrix to its Jordan form and applying problems 1154, 1156, 1153:

1162. A^{100}, where $A = \begin{pmatrix} 0 & 2 \\ -3 & 5 \end{pmatrix}$.

1163. A^{50}, where $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

1164. $\sqrt{A}$, where $A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$.

1165. $\sqrt{A}$, where $A = \begin{pmatrix} 6 & 2 \\ 3 & 7 \end{pmatrix}$.

1166. e^A, where $A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$.

1167. e^A, where $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$.

1168. e^A, where $A = \begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}$.

1169. $\ln A$, where $A = \begin{pmatrix} 4 & -15 & 6 \\ 1 & -4 & 2 \\ 1 & -5 & 3 \end{pmatrix}$.

1170. $\sin A$, where $A = \begin{pmatrix} \pi - 1 & 1 \\ -1 & \pi + 1 \end{pmatrix}$.

*1171. Prove that the equation $\sin 2A = 2 \sin A \cos A$ holds for any square matrix A.

*1172. Prove that the matrix e^A exists and is nonsingular for any square matrix A.

1173. Find the determinant of the matrix e^A, where A is a square matrix of order n.

*1174. Suppose the function $f(\lambda)$ is meaningful when $\lambda = A$. Prove that the determinant of the matrix $f(A)$ satisfies the equation $|f(A)| = f(\lambda_1) f(\lambda_2) \ldots f(\lambda_n)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix A (with account taken of multiplicity).

Sec. 15. Quadratic Forms *

This section includes problems involving quadratic forms and also problems dealing with the properties of symmetric and orthogonal matrices that are connected with the theory of quadratic forms. We will use the following terminology: a *linear transformation* is a transformation of the *unknowns*

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* Problems involving bilinear and quadratic functions are given in Sec. 24.

of the form

$$\begin{aligned} x_1 &= q_{11}y_1 + q_{12}y_2 + \ldots + q_{1n}y_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= q_{n1}y_1 + q_{n2}y_2 + \ldots + q_{nn}y_n. \end{aligned} \tag{1}$$

The matrix

$$Q = \begin{pmatrix} q_{11} & q_{12} & \ldots & q_{1n} \\ q_{21} & q_{22} & \ldots & q_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ q_{n1} & q_{n2} & \ldots & q_{nn} \end{pmatrix}, \tag{2}$$

made up of the coefficients of the transformation (1) in the appropriate order is called the matrix of that transformation. A linear transformation is said to be *nonsingular* if its matrix is nonsingular. Two quadratic forms are said to be *equivalent* if one of them can be carried into the other via a nonsingular linear transformation. The *canonical form* of a given quadratic form is the equivalent form such that does not contain products of the unknowns; the *normal* form is a canonical form in which the coefficients of the squares of the unknowns (not counting such that are zero) are equal to ± 1 for the real domain and $+1$ for the complex domain.

Find the normal form of the following quadratic forms in the field of real numbers:

1175. $x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$.

1176. $x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$.

1177. $x_1^2 - 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3$.

1178. $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$.

1179. $x_1^2 + 2x_2^2 + x_4^2 + 4x_1x_2 + 4x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4$.

Find the normal form and the nonsingular linear transformation that reduces to this form with respect to the following quadratic forms (the answer may differ from that given below; this is due to the ambiguous nature of the desired linear transformation):

1180. $x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3$.

1181. $4x_1^2 + x_2^2 + x_3^2 - 4x_1x_2 + 4x_1x_3 - 3x_2x_3$.

1182. $x_1x_2 + x_1x_3 + x_2x_3$.

1183. $2x_1^2 + 18x_2^2 + 8x_3^2 - 12x_1x_2 + 8x_1x_3 - 27x_2x_3$.

1184. $-12x_1^2 - 3x_2^2 - 12x_3^2 + 12x_1x_2 - 24x_1x_3 + 8x_2x_3$.

1185. $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$.

1186. $3x_1^2 + 2x_2^2 - x_3^2 - 2x_4^2 + 2x_1x_2 - 4x_2x_3 + 2x_2x_4$.

Reduce the following quadratic forms to canonical form with integral coefficients by means of a nonsingular linear transformation with rational coefficients and find the expression of the new unknowns in terms of the old unknowns:

1187. $2x_1^2 + 3x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 3x_2x_3$.

1188. $3x_1^2 - 2x_2^2 + 2x_3^2 + 4x_1x_2 - 3x_1x_3 - x_2x_3$.

1189. $\frac{1}{2}x_1^2 + 2x_2^2 + 3x_4^2 - x_1x_2 + x_2x_3 - x_3x_4$.

Relative to the following quadratic forms, find the nonsingular linear transformation that carries the form f into the form g (the sought-for transformation is not defined uniquely):

1190. $f = 2x_1^2 + 9x_2^2 + 3x_3^2 + 8x_1x_2 - 4x_1x_3 - 10x_2x_3$;
$g = 2y_1^2 + 3y_2^2 + 6y_3^2 - 4y_1y_2 - 4y_1y_3 + 8y_2y_3$.

1191. $f = 3x_1^2 + 10x_2^2 + 25x_3^2 - 12x_1x_2 - 18x_1x_3 + 40x_2x_3$;
$g = 5y_1^2 + 6y_2^2 + 12y_1y_2$.

1192. $f = 5x_1^2 + 5x_2^2 + 2x_3^2 + 8x_1x_2 + 6x_1x_3 + 6x_2x_3$;
$g = 4y_1^2 + y_2^2 + 9y_3^2 - 12y_1y_3$.

Bring the following quadratic forms to canonical form and find the expression for the new unknowns in terms of the old ones (the answer is not unique):

1193. $\sum_{i,j=1}^{n} a_ia_jx_ix_j$, where not all the numbers $a_1, a_2, \ldots, a_n$ are zero.

1194. $\sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} x_ix_j$. ***1195.** $\sum_{i<j}^{n} x_ix_j$. **1196.** $\sum_{i=1}^{n-1} x_ix_{i+1}$.

***1197.** $\sum_{i=1}^{n} (x_i - s)^2$, where $s = \frac{x_1 + x_2 + \ldots + x_n}{n}$.

***1198.** $\sum_{i<j}^{n} |i-j| \cdot x_ix_j$.

*1199. Given the following quadratic form:

$$f = l_1^2 + l_2^2 + \ldots + l_p^2 - l_{p+1}^2 - l_{p+2}^2 - \ldots - l_{p+q}^2,$$

where $l_1, l_2, \ldots, l_p, l_{p+1}, l_{p+2}, \ldots, l_{p+q}$ are real linear forms in $x_1, x_2, \ldots, x_n$. Prove that the positive index of inertia (that is, the number of positive coefficients in the canonical form) of the form f does not exceed p, and the negative index of inertia does not exceed q.

*1200. Prove that if it is possible to pass from each of two forms f and g to the other by means of some (not necessarily nonsingular) linear transformation, then these forms are equivalent.

Determine which of the following forms are equivalent forms in the field of real numbers:

1201. $f_1 = x_1^2 - x_1x_3$; $f_2 = y_1y_2 - y_3^2$; $f_3 = z_1z_2 + z_3^2$.

1202. $f_1 = x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3$;
$f_2 = y_1^2 + 2y_2^2 - y_3^2 + 4y_1y_2 - 2y_1y_3 - 4y_2y_3$;
$f_3 = -4z_1^2 - z_2^2 - z_3^2 - 4z_1z_2 + 4z_1z_3 + 18z_2z_3$.

1203. Show that all quadratic forms in n unknowns may be split up into classes so that two forms will be equivalent if and only if they belong to one and the same class. Find the number of such classes in the complex and real domains.

1204. What values of rank and signature characterize those classes of real equivalent quadratic forms for which the form f is equivalent to the form $-f$.

1205. Find the number of equivalence classes (in the domain of real numbers) of forms in n unknowns having a given signature s.

1206. Prove that for a quadratic form to be decomposed into a product of two linear forms it is necessary and sufficient that the following conditions hold: (a) in the domain of real numbers: the rank does not exceed two, and for a rank of two the signature is zero; (b) in the domain of complex numbers: the rank does not exceed two.

1207. Prove that a quadratic form f is positive definite if and only if the matrix can be represented in the form $A = C'C$, where C is a nonsingular real matrix and C' is the transpose of C.

1208. Use problems 913, 951 and 1207 to prove that a quadratic form is positive definite if and only if all its

corner minors are positive. A corner minor of a quadratic form is a kth-order minor occupying the first k rows and the first k columns of the matrix ($k = 1, 2, \ldots, n$; n is the order of the matrix).

1209. Prove that in a positive definite form all coefficients of the square of the unknowns are positive and that this condition is not sufficient for positive definiteness of the form.

***1210.** Prove the following assertions:

(a) For a quadratic form f to be positive definite, it is necessary and sufficient that all the principal minors (and not only the corner minors—see problem 1208) of the matrix be positive.

(b) For a quadratic form f to be nonnegative (that is, $f \geqslant 0$ for all real values of the unknowns), it is necessary and sufficient that all principal minors of the matrix be nonnegative. Use examples to show that (unlike positive definite forms) it is not sufficient for the nonnegativeness of f that all corner minors be nonnegative.

(c) For a real symmetric matrix A to be represented in the form $A = C'C$, where C is a real nonsingular matrix, it is necessary and sufficient for all corner minors of the matrix A to be positive.

(d) For a real symmetric matrix A to be represented in the form $A = C'C$, where C is a real square matrix, it is necessary and sufficient that all principal minors of A be nonnegative. Besides, if the rank of A is equal to r, then the rank of C is also equal to r, and we can assume the first r rows of C to be linearly independent and the remaining to be zero.

1211. Prove that a quadratic form f is negative definite (i.e. $f < 0$ for all real values of the unknowns, not all of which are zero) if and only if the signs of the corner minors $D_1, D_2, \ldots, D_n$ alternate and $D_1 < 0$. Here, D_k is the kth-order corner minor ($k = 1, 2, \ldots, n$).

Find all the values of the parameter λ for which the following quadratic forms are positive definite:

1212. $5x_1^2 + x_2^2 + \lambda x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$.

1213. $2x_1^2 + x_2^2 + 3x_3^2 + 2\lambda x_1x_2 + 2x_1x_3$.

1214. $x_1^2 + x_2^2 + 5x_3^2 + 2\lambda x_1x_2 - 2x_1x_3 + 4x_2x_3$.

1215. $x_1^2 + 4x_2^2 + x_3^2 + 2\lambda x_1x_3 + 10x_1x_3 + 6x_2x_3$.

1216. $2x_1^2 + 2x_2^2 + x_3^2 + 2\lambda x_1x_2 + 6x_1x_3 + 2x_2x_3$.

***1217.** The *discriminant* D_f *of a quadratic form* f is the determinant of its matrix. Prove that if we add to a positive definite quadratic form $f(x_1, x_2, \ldots, x_n)$ the square of the nonzero linear form in the same unknowns, then the discriminant of the form will increase.

***1218.** Let $f(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ be a positive definite quadratic form and let $\varphi(x_2, x_3, \ldots, x_n) = f(0, x_2, \ldots, x_n)$. Prove that the inequality $D_f \leqslant a_{11}D_\varphi$ holds for the discriminants of these forms.

***1219.** Prove that if a nonnegative quadratic form vanishes for even one nonzero set of real values of the unknowns, then that form is degenerate (that is, its discriminant is equal to zero).

***1220.** We use the term *composition of two quadratic forms*

$$f = \sum_{i,j=1}^{n} a_{ij}x_ix_j \quad \text{and} \quad g = \sum_{i,j=1}^{n} b_{ij}x_ix_j$$

for the quadratic form $(f, g) = \sum_{i,j=1}^{n} a_{ij}b_{ij}x_ix_j$.

Prove that

(a) if the forms f and g are nonnegative, then the form (f, g) is nonnegative as well;

(b) if the forms f and g are positive definite, then the form (f, g) is positive definite too.

***1221.** The term triangular transformation is used for a linear transformation of the form

$$\begin{aligned} y_1 &= x_1 + c_{12}x_2 + \ldots + c_{1n}x_n, \\ y_2 &= x_2 + c_{23}x_3 + \ldots + c_{2n}x_n, \\ &\ldots\ldots\ldots\ldots\ldots \\ y_n &= x_n. \end{aligned}$$

Prove that

(a) a triangular transformation is nonsingular and the inverse of a triangular transformation is triangular;

(b) the corner minors D_k $(k = 1, 2, \ldots, n)$ (see problem 1208) of a quadratic form $f = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ remain unaltered under a triangular transformation.

*1222. Prove that

(a) for the quadratic form of rank r, $f = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ to be brought to the form

$$f = \lambda_1 y_1^2 + \ldots + \lambda_r y_r^2, \tag{1}$$

where $\lambda_k \neq 0$ $(k = 1, 2, \ldots, r)$, by a triangular transformation, it is necessary and sufficient that

$$D_k \neq 0 \quad (k \leqslant r), \quad D_k = 0 \quad (k > r), \tag{2}$$

where D_k $(k = 1, 2, \ldots, n)$ are corner minors of the form f (see problem 1208);

(b) the indicated canonical form (1) is defined uniquely and its coefficients can be found from the formulas

$$\lambda_k = \frac{D_k}{D_{k-1}} \quad (k = 1, 2, \ldots, r;\ D_0 = 1) \tag{3}$$

(Sylvester's theorem).

1223. Let the corner minors of the quadratic form f of rank r satisfy the conditions (2) of the preceding problem. Prove that the positive index of inertia of this form is equal to the number of preservations of sign, and the negative index is equal to the number of changes of sign in the following set of numbers:

$$1 = D_0, D_1, D_2, \ldots, D_r.$$

Determine that there is one positive definite form in the following pairs of quadratic forms; find the nonsingular linear transformation that brings that form to the normal form and that brings the other form of the pair to canonical form; then write the canonical form (the linear transformation is not defined uniquely):

1224. $f = -4x_1x_2$,
$g = x_1^2 - 2x_1x_2 + 4x_2^2$.

1225. $f = x_1^2 + 26x_2^2 + 10x_1x_2$,
$g = x_1^2 + 56x_2^2 + 16x_1x_2$.

1226. $f = 8x_1^2 - 28x_2^2 + 14x_3^2 + 16x_1x_2 + 14x_1x_3 + 32x_2x_3$,
$g = x_1^2 + 4x_2^2 + 2x_3^2 + 2x_1x_3$.

1227. $f = 2x_1^2 + x_1x_2 + x_1x_3 - 2x_2x_3 + 2x_2x_4$,

$g = \frac{1}{4}x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 2x_2x_4$.

1228. $f = x_1^2 + \frac{3}{2}x_2^2 - 2x_3^2 + x_4^2 + 2x_1x_2 + 4x_1x_3$,

$g = x_1^2 + \frac{5}{4}x_2^2 + x_3^2 + x_4^2 + 2x_2x_3$.

1229. $f = x_1^2 - 15x_2^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$,

$g = x_1^2 + 17x_2^2 + 3x_3^2 + 4x_1x_2 - 2x_1x_3 - 14x_2x_3$.

***1230.** Suppose we have a pair of forms, f and g, in the same unknowns, and $g > 0$. Prove that the canonical form

$$f = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2$$

obtained for f under any linear transformation that brings g to the normal form (that is, to a sum of squares) is determined uniquely up to the order of the terms, and the coefficients λ_1, λ_2, . . ., λ_n are roots of the so-called λ-equation of the pair of forms f, g, namely, of the equation $|A - \lambda B| = 0$, where A and B are the respective matrices of the forms f and g.

Is it possible to bring the following pairs of quadratic forms to canonical form by means of a single real nonsingular linear transformation?

1231. $f = x_1^2 + 4x_1x_2 - x_2^2$,

$g = x_1^2 + 6x_1x_2 + 5x_2^2$.

1232. $f = x_1^2 + x_1x_2 - x_2^2$,

$g = x_1^2 - 2x_1x_2$.

1233. Suppose we have two positive definite forms f and g and suppose one nonsingular linear transformation of the unknowns brings the form f to $\sum_{i=1}^{n} \lambda_i y_i^2$ and the form g to the normal form, and the second transformation does the opposite, bringing the form f to the normal form and the form g to $\sum_{i=1}^{n} \mu_i z_i^2$. Find the relationship between the coefficients λ_1, . . ., λ_n and μ_1, . . ., μ_n.

Without seeking the linear transformation, find the canonical form of the given form f to which it will be carried by the transformation that brings the other given form $g>0$ to the normal form:

1234. $f = 21x_1^2 - 18x_2^2 + 6x_3^2 + 4x_1x_2 + 28x_1x_3 + 6x_2x_3,$

$g = 11x_1^2 + 6x_2^2 + 6x_3^2 - 12x_1x_2 + 12x_1x_3 - 6x_2x_3.$

1235. $f = 14x_1^2 - 4x_2^2 + 17x_3^2 + 8x_1x_2 - 40x_1x_3 - 26x_2x_3,$

$g = 9x_1^2 + 6x_2^2 + 6x_3^2 + 12x_1x_2 - 10x_1x_3 - 2x_2x_3.$

1236. Prove that the two pairs of forms f_1, g_1 and f_2, g_2, where g_1 and g_2 are positive definite, are equivalent (i.e. there exists a nonsingular linear transformation that carries f_1 into f_2 and g_1 into g_2) if and only if the roots of their λ-equations $|A_1 - \lambda B_1| = 0$ and $|A_2 - \lambda B_2| = 0$ coincide.

Determine whether the following pairs of forms are equivalent without finding the linear transformation of one pair into the other:

1237. $f_1 = 2x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_2 + 2x_1x_3,$

$g_1 = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_3,$

$f_2 = 2x_1^2 + 5x_2^2 + 2x_3^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3,$

$g_2 = x_1^2 + 6x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3.$

1238. $f_1 = 4x_1^2 + 6x_2^2 + 21x_3^2 + 4x_1x_2 - 4x_1x_3 - 22x_2x_3,$

$g_1 = 4x_1^2 + 3x_2^2 + 6x_3^2 + 4x_1x_2 - 4x_1x_3 - 6x_2x_3,$

$f_2 = 9x_1^2 + 6x_2^2 + 6x_3^2 - 6x_1x_2 - 6x_1x_3 \quad 12x_2x_3,$

$g_2 = 9x_1^2 + 3x_2^2 + 3x_3^2 - 6x_1x_2 - 6x_1x_3 + 2x_2x_3.$

Find the nonsingular linear transformation that carries the pair of quadratic forms f_1, g_1 into the other pair f_2, g_2 (the desired transformation is not defined uniquely):

1239. $f_1 = 2x_1^2 - 7x_2^2 + 2x_1x_2,\ f_2 = -7y_1^2 - 3y_2^2 - 12y_1y_2,$

$g_1 = 2x_1^2 + 13x_2^2 + 10x_1x_2,\ g_2 = 13y_1^2 + 25y_2^2 + 36y_1y_2.$

1240. $f_1 = 3x_1^2 + 2x_2^2 - 10x_1x_2,\ f_2 = -9y_1^2 - 20y_2^2 - 44y_1y_2,$

$g_1 = 2x_1^2 + 5x_2^2 - 6x_1x_2,\ g_2 = 29y_1^2 + 4y_2^2 + 20y_1y_2.$

1241. Suppose $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are two quadratic forms, at least one of which is positive definite. Prove that the "surfaces" $f = 1$ and $g = 1$ in n-dimensional space do not intersect (that is, do not have any points in common) if and only if the form $f - g$ is definite.

***1242.** Prove that the canonical form $\sum_{i=1}^{n} \lambda_i y_i^2$ to which the quadratic form f is reduced by an orthogonal transformation is uniquely defined and its coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are roots of the characteristic equation $|A - \lambda E| = 0$ of the matrix A of the form f.

Find the canonical form to which the following quadratic forms can be reduced by means of an orthogonal transformation (do not seek the transformation itself):

1243. $3x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3$.

1244. $7x_1^2 + 7x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

1245. $x_1^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$.

1246. $3x_1^2 + 3x_2^2 - x_3^2 - 6x_1x_3 + 4x_2x_3$.

***1247.** $\sum_{k=1}^{n-1} x_k x_{k+1}$.

Find the orthogonal transformation that brings the following forms to canonical form (reduction to principal axes) and write that canonical form (the transformation is not defined uniquely):

1248. $6x_1^2 + 5x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_1x_3$.

1249. $11x_1^2 + 5x_2^2 + 2x_3^2 + 16x_1x_2 + 4x_1x_3 - 20x_2x_3$.

1250. $x_1^2 + x_2^2 + 5x_3^2 - 6x_1x_2 - 2x_1x_3 + 2x_2x_3$.

1251. $x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$.

1252. $17x_1^2 + 14x_2^2 + 14x_3^2 - 4x_1x_2 - 4x_1x_3 - 8x_2x_3$.

1253. $x_1^2 - 5x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$.

1254. $8x_1^2 - 7x_2^2 + 8x_3^2 + 8x_1x_2 - 2x_1x_3 + 8x_2x_3$.

1255. $2x_1x_2 - 6x_1x_3 - 6x_2x_4 + 2x_3x_4$.

1256. $5x_1^2+5x_2^2+5x_3^2+5x_4^2-10x_1x_2+2x_1x_3+6x_1x_4+6x_2x_3+2x_2x_4-10x_3x_4$.

1257. $3x_1^2+8x_1x_2-3x_2^2+4x_3^2-4x_3x_4+x_4^2$.

1258. $x_1^2+2x_1x_2+x_2^2-2x_3^2-4x_3x_4-2x_4^2$.

1259. $9x_1^2+5x_2^2+5x_3^2+8x_4^2+8x_2x_3-4x_2x_4+4x_3x_4$.

1260. $4x_1^2-4x_1x_2+x_2^2+5x_3^2-4x_4^2+12x_4x_5+x_5^2$.

1261. $4x_1^2-4x_2^2-8x_2x_3+2x_3^2-5x_4^2+6x_4x_5+3x_5^2$.

1262. $3x_1^2+8x_1x_2-3x_2^2+4x_3^2-6x_3x_4-4x_4^2+4x_5^2+4x_5x_6+x_6^2$.

Find the canonical form to which the following forms can be brought by an orthogonal transformation and express the new unknowns in terms of the old unknowns (the sought-for transformation is not defined uniquely):

1263. $\sum_{i=1}^{n} x_i^2+\sum_{i<j}^{n} x_ix_j$. **1264.** $\sum_{i<j}^{n} x_ix_j$.

***1265.** We will say that two quadratic forms are *orthogonally equivalent* if we can pass from one to the other via an orthogonal transformation. Prove that for two forms to be orthogonally equivalent it is necessary and sufficient that the characteristic polynomials of their matrices coincide.

Determine which of the following quadratic forms are orthogonally equivalent:

1266. $f=9x_2^2+9x_3^2+12x_1x_2+12x_1x_3-6x_2x_3$;

$g=-3y_1^2+6y_2^2+6y_3^2-12y_1y_2+12y_1y_3+6y_2y_3$;

$h=11z_1^2-4z_2^2+11z_3^2+8z_1z_2-2z_1z_3+8z_2z_3$.

1267. $f=7x_1^2+x_2^2+x_3^2-8x_1x_2-8x_1x_3-16x_2x_3$;

$g=\frac{2}{3}y_1^2-\frac{1}{3}y_2^2-\frac{1}{3}y_3^2-\frac{4}{3}y_1y_2+\frac{4}{3}y_1y_3+\frac{8}{3}y_2y_3$;

$h=z_2^2-z_3^2+2\sqrt{2}z_1z_3$.

1268. Prove that any real symmetric matrix A may be represented as $A=Q^{-1}BQ$, where Q is an orthogonal matrix and B is a real diagonal matrix.

For the following matrices, find the orthogonal matrix Q and the diagonal matrix B such that the given matrix is represented as $Q^{-1}BQ$:

1269. $\begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$. **1270.** $\begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$.

***1271.** Prove that all the eigenvalues of a real symmetric matrix A lie on the interval $[a, b]$ if and only if the quadratic form with matrix $A - \lambda_0 E$ is positive definite for any $\lambda_0 < a$ and is negative definite for any $\lambda_0 > b$.

***1272.** Let A and B be real symmetric matrices. Prove that if the eigenvalues of A lie on the interval $[a, b]$ and the eigenvalues of B lie on the interval $[c, d]$, then the eigenvalues of the matrix $A + B$ lie on the interval $[a + c, b + d]$.

1273. Prove that a nonsingular quadratic form can be brought to normal form by an orthogonal transformation if and only if the matrix is orthogonal.

1274. Prove that the matrix of a positive definite quadratic form is orthogonal if and only if that form is a sum of squares. Formulate this statement in the language of matrices.

***1275.** Prove that any real nonsingular matrix A can be represented in the form $A = QB$, where Q is an orthogonal matrix and B is a triangular matrix that looks like

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & 0 & b_{33} & \dots & b_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_{nn} \end{pmatrix}$$

with positive elements on the main diagonal; prove that this representation is unique.

***1276.** Prove that

(a) any real nonsingular matrix A may be represented as $A = Q_1B_1$ and also as $A = B_2Q_2$, where the matrices Q_1 and Q_2 are real and orthogonal and the matrices B_1 and B_2 are real, symmetric and with positive corner minors. Each of these representations is unique;

(b) any complex nonsingular matrix A may be represented both as $A = Q_1B_1$ and as $A = B_2Q_2$, where the matrices Q_1 and Q_2 are unitary and the matrices B_1 and B_2 are Hermitian and with positive corner minors (the matrix B is said to be Hermitian if $\overline{B}' = B$). Each of these representations is unique;

(c) let A be a symmetric (or Hermitian) matrix with positive corner minors and let B be an orthogonal (respectively, unitary) matrix. Prove that

(1) the products AB and BA are symmetric (Hermitian) matrices with positive corner minors if and only if B is a unit matrix;

(2) the products AB and BA are orthogonal (unitary) if and only if A is a unit matrix.

Chapter IV

Vector Spaces and Their Linear Transformations

Sec. 16. Affine Vector Spaces

The following notation will be used here: vectors are denoted by lower-case boldface Latin letters, vector spaces and their subspaces and linear manifolds are denoted by upper-case boldface Latin letters. In ordinary notation, the coordinates of a vector are written out linearly in parentheses; for example, $x = (x_1, x_2, \ldots, x_n)$. In matrix notation, the vectors of the basis are written linearly in parentheses, while the coordinates of the vectors are written in a column in parentheses.

The *transition matrix* from an old basis $e_1, e_2, \ldots, e_n$ to a new basis $e'_1, e'_2, \ldots, e'_n$ is the matrix $T = (t_{ij})_1^n$ the columns of which contain the coordinates of the new basis vectors in terms of the old basis. Thus, the old and new bases are connected by the matrix equation

$$(e'_1, e'_2, \ldots, e'_n) = (e_1, e_2, \ldots, e_n) \cdot T. \qquad (1)$$

In this notation, the coordinates $x_1, x_2, \ldots, x_n$ of the vector x in the old basis are connected with the coordinates $x'_1, x'_2, \ldots, x'_n$ of the same vector in the new basis by the equations $x_i = \sum_{j=1}^{n} t_{ij} x'_j$ or, in matrix notation, by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = T \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}. \qquad (2)$$

A *linear subspace* of a vector space R is a nonempty set L (which means it contains at least one vector) of vectors in R that have the following properties:

(1) the sum $x + y$ of any two vectors in L lies in L.

(2) the product $\alpha \cdot x$ of any vector x in L by any scalar α lies in L.

A *linear manifold* of a vector space R is a collection P of vectors in R obtained by adjoining the same vector x_0 to all vectors of some subspace L of R. This relationship between L and P will be denoted thus: $P = L + x_0$ or $L = P - x_0$. We will say that the linear manifold P has been obtained from the linear subspace L by a parallel shift by the amount of the vector x_0.

The *dimension* of a linear manifold is the dimension of that linear subspace whose parallel translation yields the given manifold. The correctness of this definition follows from the assertion of problem 1331. One-dimensional linear manifolds will be called straight lines, two-dimensional ones will be termed planes.

The *sum* of two linear subspaces L_1 and L_2 of a vector space R is the collection $S = L_1 + L_2$ of all vectors of R, each of which is represented in the form $x = x_1 + x_2$, where $x_1 \in L_1$ and $x_2 \in L_2$. Here, the notation $a \in A$ means that the element a is a member of the set A. The *intersection* of two linear subspaces L_1 and L_2 of a vector space R is the collection $D = L_1 \cap L_2$ of all vectors of R, each of which belongs both in L_1 and in L_2.

The *direct sum* of two linear subspaces L_1 and L_2 of a vector space R is the sum of these subspaces, provided that their intersection consists solely of the zero vector, that is, $L_1 \cap L_2 = 0$. In the case of a direct sum we will write $S = L_1 \dot{+} L_2$.

An n-dimensional vector space will be denoted as R_n. And, unless otherwise stated, it will be assumed that the basic field is the field of real numbers, that is, R_n consists of all n-dimensional vectors with arbitrary real coordinates.

The vectors $e_1, e_2, \ldots, e_n$ and x are specified by their components (coordinates) in terms of some basis. Show that the vectors $e_1, e_2, \ldots, e_n$ themselves form a basis, and then find the coordinates of the vector x in that basis:

1277. $e_1 = (1, 1, 1)$, $e_2 = (1, 1, 2)$, $e_3 = (1, 2, 3)$; $x = (6, 9, 14)$.

1278. $e_1 = (2, 1, -3)$, $e_2 = (3, 2, -5)$, $e_3 = (1, -1, 1)$; $x = (6, 2, -7)$.

1279. $e_1 = (1, 2, -1, -2)$, $e_2 = (2, 3, 0, -1)$, $e_3 = (1, 2, 1, 4)$, $e_4 = (1, 3, -1, 0)$; $x = (7, 14, -1, 2)$.

Prove that each of the two sets of vectors is a basis and find the relationship between the coordinates of one and the same vector in the two bases:

1280. $e_1 = (1, 2, 1)$, $e_2 = (2, 3, 3)$, $e_3 = (3, 7, 1)$; $e_1' = (3, 1, 4)$, $e_2' = (5, 2, 1)$, $e_3' = (1, 1, -6)$.

1281. $e_1 = (1, 1, 1, 1)$, $e_2 = (1, 2, 1, 1)$, $e_3 = (1, 1, 2, 1)$, $e_4 = (1, 3, 2, 3)$; $e_1' = (1, 0, 3, 3)$, $e_2' = (-2, -3, -5, -4)$, $e_3' = (2, 2, 5, 4)$, $e_4' = (-2, -3, -4, -4)$.

1282. Find the coordinates of the polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

(a) in the basis $1, x, x^2, \ldots, x^n$;

(b) in the basis $1, x - \alpha, (x - \alpha)^2, \ldots, (x - \alpha)^n$ after having ascertained that the last two polynomials do indeed form a basis.

1283. Find the transition matrix from the basis $1, x, x^2, \ldots, x^n$ to the basis $1, x - \alpha, (x - \alpha)^2, \ldots, (x-\alpha)^n$ in the space of polynomials of degree less than or equal to n.

1284. How does the transition matrix vary from one basis to another

(a) if two vectors of the first basis are interchanged?

(b) if two vectors of the second basis are interchanged?

(c) if the vectors in both bases are given in reversed order?

State whether each of the following sets of vectors is a linear subspace of the appropriate vector space (from 1285 to 1293):

1285. All vectors of an n-dimensional vector space whose coordinates are integers.

1286. All vectors of a plane, each of which lies on one of the coordinate axes Ox or Oy.

1287. All vectors of a plane, the endpoints of the vectors lying on a given straight line (unless otherwise stated, the origin of any vector is assumed to coincide with the coordinate origin).

1288. All vectors of a plane, the origin and terminus of each vector lying on a given straight line.

1289. All vectors of three-dimensional space, the termini of the vectors not lying on a given straight line.

1290. All vectors of a plane, the termini of the vectors lying in the first quadrant of the coordinate system.

1291. All vectors in R_n whose coordinates satisfy the equation $x_1 + x_2 + \ldots + x_n = 0$.

1292. All vectors in R_n, whose coordinates satisfy the equation $x_1 + x_2 + \ldots + x_n = 1$.

1293. All vectors that are linear combinations of the given vectors: $x_1, x_2, \ldots, x_k$ in R_n.

1294. Enumerate all linear subspaces of three-dimensional vector space.

1295. Suppose the linear subspace L_1 lies in the linear subspace L_2. Prove that the dimension of L_1 does not exceed the dimension of L_2, the dimensions being equal if and only if $L_1 = L_2$. Does this assertion hold true for any two linear subspaces of the given space?

1296. Prove that if the sum of the dimensions of two linear subspaces of n-dimensional space exceeds n, then these subspaces have a nonzero vector in common.

Prove that the following sets of vectors form linear subspaces and find the basis and dimension of each (1297 to 1300):

1297. All n-dimensional vectors with the first and last coordinates equal.

1298. All n-dimensional vectors whose coordinates with even-number labels are zero.

1299. All n-dimensional vectors whose coordinates with even-number labels are equal.

1300. All n-dimensional vectors of the form $(\alpha, \beta, \alpha, \beta, \alpha, \beta, \ldots)$, where α and β are any numbers.

1301. Prove that all square matrices of order n with real elements (or elements taken from any field P) form a vector space over the field of real numbers (respectively, over the field P) if the operations involved are addition of matrices and multiplication of a matrix by a scalar. Find the basis and dimension of that space.

1302. Prove that all polynomials of degree $\leqslant n$ in one unknown with real coefficients (or with coefficients from any field P) form a vector space if the operations involved are ordinary addition of polynomials and ordinary multiplication of a polynomial by a scalar. Find the basis and dimension of that space.

1303. Prove that all symmetric matrices form a linear subspace of the space of all nth-order square matrices. Find the basis and dimension of that subspace.

1304. Prove that skew-symmetric matrices form a linear subspace of all nth-order square matrices. Find the basis and dimension of that subspace.

1305. Prove that if a linear subspace L of the space of polynomials of degree $\leqslant n$ contains at least one polynomial of degree k for $k = 0, 1, 2, \ldots, p$, but does not contain polynomials of degree $k > p$, then it coincides with the subspace L_p of all polynomials of degree $\leqslant p$.

1306. Let f be a nonnegative quadratic form in n unknowns of rank r. Prove that all solutions of the equation $f = 0$ form an $(n - r)$-dimensional linear subspace of the space R_n.

1307. Prove that the solutions of any system of homogeneous linear equations in n unknowns of rank r form a linear subspace of the n-dimensional space R_n of dimension $d = n - r$ and, conversely, for any linear subspace L of dimension d of the space R_n there exists a system of homogeneous linear equations in n unknowns of rank $r = n - d$, whose solutions fill the given subspace L exactly.

1308. Find some basis and the dimension of a linear subspace L of the space R_n if L is specified by the equation $x_1 + x_2 + \ldots + x_n = 0$.

1309. Prove that the dimension of a linear subspace L spanned by the vectors $x_1, x_2, \ldots, x_k$ (that is, the subspace of all linear combinations of the given vectors) is equal to the rank of the matrix made up of the coordinates of the given vectors in some basis; for the basis of the subspace L we can take any maximal linearly independent subset of the set of given vectors.

Find the dimensions and bases of the linear subspaces spanned by the following sets of vectors:

1310. $a_1 = (1, 0, 0, -1)$, $a_2 = (2, 1, 1, 0)$, $a_3 = (1, 1, 1, 1)$, $a_4 = (1, 2, 3, 4)$, $a_5 = (0, 1, 2, 3)$.

1311. $a_1 = (1, 1, 1, 1, 0)$, $a_2 = (1, 1, -1, -1, -1)$, $a_3 = (2, 2, 0, 0, -1)$, $a_4 = (1, 1, 5, 5, 2)$, $a_5 = (1, -1, -1, 0, 0)$.

Find the systems of linear equations that specify the linear subspaces spanned by the following sets of vectors (1312-1316):

1312. $a_1 = (1, -1, 1, 0)$, $a_2 = (1, 1, 0, 1)$, $a_3 = (2, 0, 1, 1)$.

1313. $a_1 = (1, -1, 1, -1, 1)$, $a_2 = (1, 1, 0, 0, 3)$, $a_3 = (3, 1, 1, -1, 7)$, $a_4 = (0, 2, -1, 1, 2)$.

1314. Prove that the union and the intersection of two linear subspaces of the space R_n are themselves linear subspaces of that space.

1315. Prove that the union (or sum) $S = L_1 + L_2$ of two linear subspaces of the space R_n is equal to the intersection of all linear subspaces in R_n that contain both L_1 and L_2.

1316. Prove that the union of the dimensions of two linear subspaces of R_n is equal to the dimension of the union plus the dimension of the intersection of these subspaces.

Find the dimension s of the union and the dimension d of the intersection of the linear subspaces: L_1 spanned by the vectors $a_1, a_2, \ldots, a_k$ and L_2 spanned by the vectors $b_1, b_2, \ldots, b_l$:

1317. $a_1 = (1, 2, 0, 1)$, $a_2 = (1, 1, 1, 0)$ and $b_1 = (1, 0, 1, 0)$, $b_2 = (1, 3, 0, 1)$.

1318. $a_1 = (1, 1, 1, 1)$, $a_2 = (1, -1, 1, -1)$, $a_3 = (1, 3, 1, 3)$; $b_1 = (1, 2, 0, 2)$, $b_2 = (1, 2, 1, 2)$, $b_3 = (3, 1, 3, 1)$.

***1319.** Suppose L_1 is a linear subspace of the space R_n with basis $a_1, a_2, \ldots, a_k$ and let L_2 be a linear subspace of the same space with basis $b_1, b_2, \ldots, b_l$.

Prove the following rules for constructing the basis of the union $S = L_1 + L_2$ and the basis of the intersection $D = L_1 \cap L_2$:

(1) The basis of the union S is the maximum linearly independent subset of the set of vectors $a_1, \ldots, a_k, b_1, \ldots, b_l$. Constructing it reduces to computing the rank of the matrix made up of the coordinates of this last set of vectors.

(2) The basis of the union S may be obtained by adjoining to the linearly independent vectors $a_1, \ldots, a_k$ certain of the vectors $b_1, \ldots, b_l$ (see problem 659). By changing,

if necessary, the order of these vectors, we can consider that the vectors $a_1, \ldots, a_k, b_1, \ldots, b_{s-k}$ form a basis of S.

The equation

$$x_1a_1 + \ldots + x_ka_k = y_1b_1 + \ldots + y_lb_l \qquad (1)$$

is equivalent to a system of n homogeneous linear equations in $k + l$ unknowns $x_1, \ldots, x_k, y_1, \ldots, y_l$ of rank s. Since the first s columns of the matrix of the system are linearly independent and, hence, at least one minor of order s in these columns is different from zero, we can take the last $k + l - s = d$ unknowns $y_{s-k+1}, \ldots, y_l$ for the free unknowns. We can therefore find, for the system of equations (1), the following fundamental set of solutions

$$x_{i1}, x_{i2}, \ldots, x_{ik}, y_{i1}, y_{i2}, \ldots, y_{il} \quad (i = 1, 2, \ldots, d) \qquad (2)$$

such that

$$\begin{vmatrix} y_{1,\,s-k+1} & \cdots & y_{1,\,l} \\ y_{d,\,s-k+1} & \cdots & y_{d,\,l} \end{vmatrix} \neq 0. \qquad (3)$$

Then the basis of the intersection D is the set of vectors

$$c_i = \sum_{j=1}^{l} y_{ij}b_j \quad (i = 1, 2, \ldots, d). \qquad (4)$$

Note. Since $d = k + l - s$, this yields the second solution of problem 1316.

Find the bases of the unions and intersections of linear subspaces spanned by the sets of vectors $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$:

1320. $a_1 = (1, 2, 1)$, $a_2 = (1, 1, -1)$, $a_3 = (1, 3, 3)$; $b_1 = (2, 3, -1)$, $b_2 = (1, 2, 2)$, $b_3 = (1, 1, -3)$.

1321. $a_1 = (1, 2, 1, -2)$, $a_2 = (2, 3, 1, 0)$, $a_3 = (1, 2, 2, -3)$; $b_1 = (1, 1, 1, 1)$, $b_2 = (1, 0, 1, -1)$, $b_3 = (1, 3, 0, -4)$.

1322. $a_1 = (1, 1, 0, 0)$, $a_2 = (0, 1, 1, 0)$, $a_3 = (0, 0, 1, 1)$; $b_1 = (1, 0, 1, 0)$, $b_2 = (0, 2, 1, 1)$, $b_3 = (1, 2, 1, 2)$.

1323. Prove that if the dimension of the union of two linear subspaces of the space R_n is one unit greater than the dimension of their intersection, then the union coincides with one of the subspaces and the intersection coincides with the other.

1324. Let L, L_1, L_2 be linear subspaces of the space R_n. Prove that L is a direct sum of L_1, L_2 if and only if the following conditions hold true:

(a) L contains L_1 and L_2;

(b) every vector $x \in L$ is uniquely represented as $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. In other words, the sum $L = L_1 + L_2$ is a direct sum if and only if for every vector $x \in L$ the representation $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$, is unique.

1325. Prove that the sum S of linear subspaces L_1 and L_2 is a direct sum if and only if at least one vector $x \in S$ can be uniquely represented as $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$.

1326. Suppose a linear subspace L is a direct sum of the linear subspaces L_1 and L_2. Prove that the dimension of L is equal to the sum of the dimensions of L_1 and L_2; also, any bases of L_1 and L_2 together yield a basis of L.

1327. Prove that for any linear subspace L_1 of the space R_n it is possible to find another subspace L_2 such that the whole space R_n will be a direct sum of L_1 and L_2.

1328. Prove that the space R_n is a direct sum of two linear subspaces: L_1 specified by the equation $x_1 + x_2 + \ldots + x_n = 0$ and L_2 given by the system of equations $x_1 = x_2 = \ldots = x_n$. Find the projections of the unit vectors

$$e_1 = (1, 0, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0),\ \ldots$$
$$\ldots,\ e_n = (0, 0, 0, \ldots, 1)$$

on L_1 parallel to L_2 and on L_2 parallel to L_1.

1329. Prove that the space of all square matrices of order n is a direct sum of the linear subspaces L_1 of symmetric matrices and L_2 of skew-symmetric matrices. Find the projections A_1 and A_2 of the matrix

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

on L_1 parallel to L_2 and on L_2 parallel to L_1.

1330. Prove that the solutions of any consistent system of linear equations in n unknowns of rank r form a linear manifold of the space R_n of dimension $d = n - r$ and,

conversely, for any d-dimensional linear manifold P of the space R_n there exists a system of linear equations in n unknowns of rank $r = n - d$, the solutions of which exactly fill the given manifold P.

1331. Given two linear manifolds (see Sec. 16, introduction) $P_1 = L_1 + x_1$ and $P_2 = L_2 + x_2$, where L_1, L_2 are linear subspaces and x_1, x_2 are vectors of the space R_n. Prove that $P_1 = P_2$ if and only if $L_1 = L_2$ and $x_1 - x_2 \in L_1$. Thus, the linear space whose parallel translation yields the given manifold is defined uniquely.

1332. Prove that if $P = L + x_0$, where L is a linear subspace and x_0 is a vector of the space R_n, then the vector x_0 belongs to the manifold P and replacing this vector by any other vector $x \in P$ yields the same manifold P.

1333. Prove that if a straight line has two common points with a linear manifold, then it lies entirely in that manifold (here, a point is identified with a vector having the same coordinates, i.e. issuing from the coordinate origin to the given point).

***1334.** Prove that any two straight lines of the space R_n ($n \geqslant 3$) are contained in a certain three-dimensional linear manifold lying in R_n.

1335. Find the conditions that are necessary and sufficient for the two straight lines $x = a_0 + a_1 t$ and $x = b_0 + b_1 t$ of the space R_n ($n > 1$) to lie in a single plane.

1336. Find the necessary and sufficient conditions for two straight lines $x = a_0 + a_1 t$ and $x = b_0 + b_1 t$ to pass through a single point without being coincident. Indicate a method for finding the point of intersection of these straight lines.

Find the point of intersection of the two straight lines $a_0 + a_1 t$ and $b_0 + b_1 t$:

1337. $a_0 = (2, 1, 1, 3, -3)$, $a_1 = (2, 3, 1, 1, -1)$; $b_0 = (1, 1, 2, 1, 2)$, $b_1 = (1, 2, 1, 0, 1)$.

1338. $a_0 = (3, 1, 2, 1, 3)$, $a_1 = (1, 0, 1, 1, 2)$; $b_0 = (2, 2, -1, -1, -2)$; $b_1 = (2, 1, 0, 1, 1)$.

1339. Find the necessary and sufficient conditions for drawing through a point specified by a vector c a unique straight line intersecting two given straight lines $x = a_0 + a_1 t$ and $x = b_0 + b_1 t$. Indicate a method for constructing

such a straight line and the points of its intersection with the given straight lines.

Find a straight line that passes through a point specified by a vector c and intersects the straight lines $x = a_0 + a_1 t$, $x = b_0 + b_1 t$, and find the points of intersection of the desired straight line with the two given straight lines:

1340. $a_0 = (1, 0, -2, 1)$, $a_1 = (1, 2, -1, -5)$; $b_0 = (0, 1, 1, -1)$, $b_1 = (2, 3, -2, -4)$; $c = (8, 9, -11, -15)$.

1341. $a_0 = (1, 1, 1, 1)$, $a_1 = (1, 2, 1, 0)$; $b_0 = (2, 2, 3, 1)$, $b_1 = (1, 0, 1, 3)$; $c = (4, 5, 2, 7)$.

1342. Prove that any two planes of the space R_n are contained in a linear manifold of dimension $\leqslant 5$.

1343. Prove that two linear manifolds of the space R_n of dimensions k and l are contained in a linear manifold of dimension $\leqslant k + l + 1$.

1344. Prove that if two linear manifolds—P of dimension k and Q of dimension l—of the space R_n have a common point, and $k + l > n$, then their intersection is a linear manifold of dimension $\geqslant k + l - n$. What theorems follow from this for three-dimensional and four-dimensional space?

1345. Describe all cases of mutual positions of two planes $x = a_0 + a_1 t_1 + a_2 t_2$ and $x = b_0 + b_1 t_1 + b_2 t_2$ in n-dimensional space and indicate the necessary and sufficient conditions for each of these cases.

1346. Suppose

$$a_0, a_1, \ldots, a_k \tag{1}$$

are any $k + 1$ vectors of the space R_n. Prove that all vectors of the form

$$x = \alpha_0 a_0 + \alpha_1 a_1 + \ldots + \alpha_k a_k \tag{2}$$

where the numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$ satisfy the condition

$$\alpha_0 + \alpha_1 + \ldots + \alpha_k = 1 \tag{3}$$

form a linear manifold P whose dimension is equal to the rank of the set of vectors

$$a_1 - a_0, \ldots, a_k - a_0. \tag{4}$$

P is a manifold of smallest dimension that contains all the vectors (1). The role of the vector a_0 may be played by any of these vectors. Conversely, for any k-dimensional

linear manifold P there exists a set of $k + 1$ vectors (1) such that P consists of all the vectors of the form (2) given condition (3), and the set of vectors (4) is linearly independent.

*1347. *A line segment with endpoints at points specified by the vectors a_1, a_2 of the space R_n* is defined as the collection of all points given by vectors of the form $x = \alpha_1 a_1 + \alpha_2 a_2$, where $\alpha_1 + \alpha_2 = 1$ and $0 \leqslant \alpha_1 \leqslant 1$, $0 \leqslant \alpha_2 \leqslant 1$. The set M of points of the space R_n is said to be *convex* if for any two points in M the entire segment with endpoints at these points lies in M. Show that an intersection of any set of convex sets of the space R_n is a convex set. *A convex closure of the given set A in R_n* is the intersection of all convex sets in R_n that contain A. Prove that the convex closure of a finite system of points of the space R_n specified by the vectors $a_1, a_2, \ldots, a_k$ consists of all points given by vectors of the form $x = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_k a_k$, where $\sum_{i=1}^{k} \alpha_i = 1$ and $0 \leqslant \alpha_i \leqslant 1$ $(i = 1, 2, \ldots, k)$.

*1348. Find the shape of a solid obtained in the section of a four-dimensional parallelepiped (it is a four-dimensional cube in the case of a rectangular Cartesian coordinate system) $|x_i| \leqslant 1$ $(i = 1, 2, 3, 4)$ by a three-dimensional hyperplane with the equation $x_1 + x_2 + x_3 + x_4 = 0$.

*1349. Find the projection of a four-dimensional tetrahedron bounded by coordinate three-dimensional subspaces and by the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ on the subspace $x_1 + x_2 + x_3 + x_4 = 0$ parallel to the straight line $x_1 = x_2 = x_3 = x_4$.

*1350. Prove that a diagonal of an n-dimensional parallelepiped is divided into n equal parts by the points of its intersection with $(n - 1)$-dimensional linear manifolds drawn through all vertices of the parallelepiped parallel to a linear manifold passed through the endpoints of all n edges, whose origin coincides with one of the endpoints of the given diagonal.

Sec. 17. Euclidean and Unitary Spaces

A *Euclidean* (respectively, *unitary*) space R_n is an n-dimensional vector space over the field of real numbers (respectively, complex numbers) in which each pair of vectors x, y is associated with a real (respectively, complex) number

(x, y) called the scalar product of the vectors, provided the following conditions hold:

(a) in the case of a Euclidean space:

$$(x, y) = (y, x), \tag{1}$$

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y), \tag{2}$$

$$(\alpha x, y) = \alpha (x, y) \tag{3}$$

for any real number α.

$$\text{If } x \neq 0, \text{ then } (x, x) > 0; \tag{4}$$

(b) in the case of a unitary space:

$$(x, y) = \overline{(y, x)}, \tag{1'}$$

$$(x_1 + x_2, y) = (x_1, y) + (x_2\ y), \tag{2'}$$

which coincides with (2);

$$(\alpha x, y) = \alpha (x, y) \tag{3'}$$

for any complex number α.

$$\text{If } x \neq 0, \text{ then } (x, x) > 0, \tag{4'}$$

which coincides with (4).

The basis (or, generally, any set of vectors) $e_1, e_2, \ldots, e_n$ is said to be *orthonormal* if

$$(e_i, e_j) = \begin{cases} 1 \text{ when } i = j, \\ 0 \text{ when } i \neq j. \end{cases}$$

Unless otherwise stated, the coordinates of the vectors are assumed to be taken in an orthonormal basis.

The vectors x and y are said to be orthogonal if $(x, y) = 0$.

The process of orthogonalization of a set of vectors $a_1, a_2, \ldots, a_s$ is a transition from that set to a new set $b_1, b_2, \ldots, b_s$ constructed as follows: $b_1 = a_1$; $b_k = a_k - \sum_{i=1}^{k-1} c_i b_i$ $(k = 2, 3, \ldots, s)$, where $c_i = \frac{(a_k, b_i)}{(b_i, b_i)}$ $(i = 1, 2, \ldots, k-1)$ if $b_i \neq 0$ and c_i is any number if $b_i = 0$.

The value of c_i is obtained by multiplying into b_i the equation expressing b_k in terms of a_k and b_i $(i = 1, 2, \ldots, k-1)$, provided that $(b_k, b_i) = 0$.

1351. Prove that the following properties flow from the properties of a scalar product as indicated in the introduction:

(a) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$ for any vectors of a Euclidean (or unitary) space;

(b) $(x, \alpha y) = \alpha (x, y)$ for any vectors x, y of a Euclidean space and any real number α;

(c) $(x, \alpha y) = \bar{\alpha} (x, y)$ for any vectors x, y of unitary space and any complex number α;

(d) $(x_1 - x_2, y) = (x_1, y) - (x_2, y)$;

(e) $(x, 0) = 0$.

1352. What properties must the bilinear form $g = \sum_{i, j=1}^{n} a_{ij} x_i y_j$ have so that its value in the coordinates of any two vectors

$$x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n)$$

of a real vector space R_n relative to some basis $e_1, e_2, \ldots, e_n$ may be taken for the scalar product of these vectors, which product defines an n-dimensional Euclidean space? Find the scalar products of the vectors of the chosen basis.

1353. Given a Hermitian bilinear form $g = \sum_{i, j=1}^{n} a_{ij} x_i \bar{y}_j$. The bar over the unknown y_j means that when y_j is replaced by its numerical value α_j, the $\bar{y}_j$ should be replaced by the complex conjugate value $\bar{\alpha}_j$. Let the matrix $A = (a_{ij})_1^n$ of this form be Hermitian, that is, $\bar{a}_{ij} = a_{ji}$ $(i, j = 1, 2, \ldots, n)$. Show that the values of the corresponding Hermitian quadratic form $f = \sum_{i, j=1}^{n} a_{ij} x_i \bar{x}_j$ for arbitrary complex values $x_1, x_2, \ldots, x_n$ are real, and if the form f is positive definite, that is, $f > 0$ for any complex values $x_1, x_2, \ldots, x_n$ not all zero, then specifying the scalar product by the equation $(x, y) = \sum_{i, j=1}^{n} a_{ij} x_i \bar{y}_j$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the coordinates of the vectors x and y respectively in some basis $e_1, \ldots, e_n$

of the complex vector space R_n, transforms this space into a unitary space; note that any unitary space may be obtained in this fashion.

1354. Prove that the scalar product of any two vectors

$$x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n)$$

of a Euclidean space is expressed by the equation

$$(x, y) = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

if and only if the basis in which the coordinates are taken is an orthonormal basis.

1355. Suppose L_1 and L_2 are linear subspaces of a Euclidean (or unitary) space R_n, the dimension of L_1 being less than that of L_2; prove that there is a nonzero vector in L_2 orthogonal to all vectors in L_1.

1356. Prove that any set of pairwise orthogonal nonzero vectors (in particular, any orthonormal set) is linearly independent.

Verify that the vectors of the following sets are pairwise orthogonal and complete them to form orthogonal bases:

1357. (1, −2, 2, −3), (2, −3, 2, 4).

1358. (1, 1, 1, 2), (1, 2, 3, −3).

Find the vectors that complete the following sets of vectors to form orthonormal bases:

1359. $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$.

1360. $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

Apply orthogonalization (see Sec. 17, introduction) to construct an orthogonal basis of the subspace spanned by the given set of vectors:

1361. (1, 2, 2, −1), (1, 1, −5, 3), (3, 2, 8, −7).

1362. (1, 1, −1, −2), (5, 8, −2, −3), (3, 9, 3, 8).

1363. (2, 1, 3, −1),
(7, 4, 3, −3),
(1, 1, −6, 0),
(5, 7, 7, 8).

1364. *The orthogonal complement of the subspace L of the space R_n* is the set L^* of all vectors in R_n, each of which is orthogonal to all vectors in L.

Prove that

(a) L^* is a linear subspace of the space R_n;

(b) the sum of the dimensions of L and L^* is equal to n;

(c) the space R_n is a direct sum of the subspaces L and L^*.

1365. Prove that the orthogonal complement of a linear subspace of the space R_n has the following properties:

(a) $(L^*)^* = L$, (b) $(L_1 + L_2)^* = L_1^* \cap L_2^*$

(c) $(L_1 \cap L_2)^* = L_1^* + L_2^*$, (d) $R_n^* = O$, $O^* = R_n$.

Here, O is the zero subspace containing the sole zero vector $\mathbf{0}$.

1366. Find the basis of the orthogonal complement L^* of the subspace L spanned by the vectors:

$a_1 = (1, 0, 2, 1)$, $a_2 = (2, 1, 2, 3)$, $a_3 = (0, 1, -2, 1)$.

1367. A linear subspace L is given by the following equations:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 - x_4 &= 0, \\ 3x_1 + 2x_2 - 2x_4 &= 0, \\ 3x_1 + x_2 + 9x_3 - x_4 &= 0. \end{aligned}$$

Find the equations that specify the orthogonal complement L^*.

1368. Show that specification of the linear subspace L of R_n and its orthogonal complement L^* in an orthonormal basis are related as follows: the coefficients of a linearly independent system of linear equations specifying one of these subspaces serve as the coordinates of the vectors of the basis of the other subspace.

1369. Let L be a linear subspace of the space R_n. Prove that any vector x in R_n can be uniquely represented as $x = y + z$, where y belongs to L and z is orthogonal to L. y is termed the orthogonal projection of the vector x on the subspace L, and z is the orthogonal component of x relative to L. Indicate a procedure for computing y and z.

Find the orthogonal projection y and the orthogonal component z of the vector x on the linear subspace L:

1370. $x = (4, -1, -3, 4)$. L is spanned by the vectors $a_1 = (1, 1, 1, 1)$, $a_2 = (1, 2, 2, -1)$, $a_3 = (1, 0, 0, 3)$.

1371. $x = (5, 2, -2, 2)$. L is spanned by the vectors $a_1 = (2, 1, 1, -1)$, $a_2 = (1, 1, 3, 0)$, $a_3 = (1, 2, 8, 1)$.

1372. $x = (7, -4, -1, 2)$. L is given by the system of equations

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 2x_2 + 2x_3 + x_4 &= 0, \\ x_1 + 2x_2 + 2x_3 - 9x_4 &= 0. \end{aligned}$$

***1373.** *The distance from the point given by the vector x to the linear manifold $P = L + x_0$* is the least of the distances from the given point to the points of the manifold, that is, the minimum of the lengths of the vectors $x - u$, where u is a vector of P.

Prove that this distance is equal to the length of the orthogonal component z of the vector $x - x_0$ relative to the linear subspace L, a parallel translation of which yields the manifold P.

1374. Find the distance from the point specified by the vector x up to the linear manifold specified by the following system of equations:

(a) $x = (4, 2, -5, 1)$;

$$\begin{aligned} 2x_1 - 2x_2 + x_3 + 2x_4 &= 9, \\ 2x_1 - 4x_2 + 2x_3 + 3x_4 &= 12. \end{aligned}$$

(b) $x = (2, 4, -4, 2)$;

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 1, \\ x_1 + 3x_2 + x_3 - 3x_4 &= 2. \end{aligned}$$

***1375.** Prove that the distance d from the point given by the vector x up to the linear manifold $P = L + x_0$, where L is a linear subspace with basis $a_1, a_2, \ldots, a_k$, is computed with the aid of the Gram determinant (see problem 1415) via the formula

$$d^2 = \frac{G(a_1, a_2, \ldots, a_k, x - x_0)}{G(a_1, a_2, \ldots, a_k)}.$$

*1376. The distance between two linear manifolds $P_1 = L_1 + x_1$ and $P_2 = L_2 + x_2$ is the minimum of the distances of any two points, one of which belongs to P_1 and the other to P_2. Prove that this distance is equal to the length of the orthogonal component of the vector $x_1 - x_2$ relative to the linear subspace $L = L_1 + L_2$.

1377. Find the distance between the two planes $x = a_1t_1 + a_2t_2 + x_1$ and $x = a_3t_1 + a_4t_2 + x_2$, where

$$a_1 = (1, 2, 2, 2), \quad a_2 = (2, -2, 1, 2),$$
$$a_3 = (2, 0, 2, 1), \quad a_4 = (1, -2, 0, -1);$$
$$x_1 = (4, 5, 3, 2), \quad x_2 = (1, -2, 1, -3).$$

*1378. A regular n-dimensional simplex of the Euclidean space R_p $(p \geqslant n)$ is a convex closure (see problem 1347) of a set of equally spaced points $n + 1$. The points of the set are termed vertices; the line segments joining them are called edges, and the convex closures of the subsets of $k+1$ points of the given set are termed k-dimensional faces of the simplex. Two faces are said to be opposite faces if they do not have any vertices in common and any one of the $n + 1$ vertices of the simplex is a vertex of one of the faces.

Find the distance between two opposite faces of dimensions k and $n - k - 1$ of an n-dimensional simplex with edge length unity, and prove that it is equal to the distance between the centres of the faces.

*1379. Let e be a vector of unit length of the Euclidean (or unitary) space R_n. Prove that any vector x in R_n can be uniquely represented as $x = \alpha e + z$, where $(z, e) = 0$. The number α is called the projection of the vector x on the direction of e and is denoted as $\mathrm{pr}_e x$.

Prove that

(a) $\mathrm{pr}_e (x + y) = \mathrm{pr}_e x + \mathrm{pr}_e y$;

(b) $\mathrm{pr}_e (\lambda x) = \lambda \mathrm{pr}_e x$;

(c) $\mathrm{pr}_e x = (x, e)$;

(d) for any orthonormal basis $e_1, \ldots, e_n$ and any vector x the equation $x = \sum_{i=1}^{n} (\mathrm{pr}_{e_i} x) \cdot e_i$ holds true.

*1380. Let $e_1, \ldots, e_k$ be an orthonormal set of vectors of the Euclidean space R_n. Prove that for any vector x in R_n

we have the inequality (Bessel's inequality)

$$\sum_{i=1}^{k} (\mathrm{pr}_{e_i} x)^2 \leqslant |x|^2.$$

And this inequality becomes the Parseval equality for any x in R_n when and only when $k = n$, that is, the set $e_1, \ldots, e_k$ is an orthonormal basis.

*1381. Prove the Cauchy-Bunyakovsky inequality

$$(x, y)^2 \leqslant (x, x)\,(y, y)$$

for arbitrary vectors x and y of Euclidean space, the equals sign occurring if and only if the vectors x and y are linearly dependent.

*1382. Prove the Cauchy-Bunyakovsky inequality

$$(x, y)\,(y, x) \leqslant (x, x)\,(y, y)$$

for arbitrary vectors x and y of unitary space, the equals sign occurring if and only if the vectors x and y are linearly dependent.

1383. Using the Cauchy-Bunyakovsky inequality, prove the following inequalities:

$$\text{(a)}\ \Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leqslant \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2$$

for any real numbers $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ (see problem 503);

$$\text{(b)}\ \Big|\sum_{i=1}^{n} a_i \overline{b}_i\Big|^2 \leqslant \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2$$

for any complex numbers $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ (see problem 505).

1384. Specified in an infinite-dimensional vector space of all real functions continuous on the interval $[a, b]$ with the ordinary addition of functions and multiplication of a function by a scalar is the scalar product $(f, g) = \int_a^b f(x)\, g(x)\, dx$. Verify the fulfilment of all properties of a scalar product of Euclidean space (see Sec. 17, introduction) and write down the Cauchy-Bunyakovsky inequality for that space.

Find the lengths of the sides and the interior angles of triangles whose vertices are given by their coordinates:

1385. $A(2, 4, 2, 4, 2)$; $B(6, 4, 4, 4, 6)$; $C(5, 7, 5, 7, 2)$.

1386. $A(1, 2, 3, 2, 1)$; $B(3, 4, 0, 4, 3)$; $C\left(1+\frac{5}{26}\sqrt{78},\right.$ $\left.2+\frac{5}{13}\sqrt{78},\ 3+\frac{10}{13}\sqrt{78},\ 2+\frac{5}{13}\sqrt{78},\ 1+\frac{5}{26}\sqrt{78}\right)$.

1387. Prove the following generalization of the theorem of elementary mathematics concerning two perpendicular lines: if the vector x of a Euclidean (or unitary) space is orthogonal to each of the vectors $a_1, a_2, \ldots, a_k$, then it is orthogonal to each vector of the linear subspace L spanned by the vectors $a_1, a_2, \ldots, a_k$.

1388. Prove that if $x = \alpha y$, then $|x| = |\alpha| \cdot |y|$. Here, $|x|$, $|y|$ are the lengths of the vectors x and y.

***1389.** Prove that the square of the diagonal of a rectangular n-dimensional parallelepiped is equal to the sum of the squares of its edges emanating from a single vertex (this is an n-dimensional generalization of the Pythagoras theorem).

***1390.** Prove a theorem stating that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

1391. Prove a theorem stating that the square of a side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of those sides into the cosine of the angle between them. Do this by using the scalar multiplication of vectors.

1392. Use the Cauchy-Bunyakovsky inequality to prove the triangle inequality: if $\rho(X, Y)$ is the distance between the points X and Y, then for any three points A, B and C we have $\rho(A, B) + \rho(B, C) \geqslant \rho(A, C)$, equality occurring if and only if the vector x drawn from A to B and the vector y drawn from B to C are collinear and in the same direction.

1393. Find the number of diagonals of an n-dimensional cube, which diagonals are orthogonal to a given diagonal.

1394. Find the length of a diagonal of an n-dimensional cube with edge a and find the limit of that length as $n \to \infty$.

1395. Prove that all diagonals of an n-dimensional cube

form the same angle φ_n with all the edges. Find that angle and also its limit as $n \to \infty$. For what value of n do we obtain $\varphi_n = 60°$?

1396. Find an expression for the radius R of a sphere described about an n-dimensional cube using the edge a; which of the quantities R and a is greater for different values of n?

1397. Prove that the orthogonal projection of any edge of an n-dimensional cube on any diagonal of that cube is equal, in absolute value, to $\frac{1}{n}$ of the length of the diagonal.

***1398.** Prove that the orthogonal projections of the vertices of an n-dimensional cube on any diagonal divides it into n equal parts.

***1399.** Let x and y be nonzero vectors of the Euclidean space R_n. Prove that

(a) $x = \alpha y$, where $\alpha > 0$ if and only if the angle between x and y is zero;

(b) $x = \alpha y$, where $\alpha < 0$ if and only if the angle between x and y is equal to π.

***1400.** Prove that of all vectors of the linear subspace L the smallest angle with the given vector x is formed by the orthogonal projection y of the vector x on L. Here, the equality $\cos(x, y) = \cos(x, y')$, where $y' \in L$, holds if and only if $y' = \alpha y$, where $\alpha > 0$.

1401. Find the angle between the diagonal of an n-dimensional cube and its k-dimensional face.

Find the angle between the vector x and a linear subspace spanned by the vectors $a_1, a_2, \ldots, a_k$:

1402. $x = (2, 2, 1, 1)$;
$a_1 = (3, 4, -4, -1)$;
$a_2 = (0, 1, -1, 2)$.

1403. $x = (1, 0, 3, 0)$;
$a_1 = (5, 3, 4, -3)$;
$a_2 = (1, 1, 4, 5)$;
$a_3 = (2, -1, 1, 2)$.

1404. The term *angle* between two linear subspaces L_1 and L_2 of the Euclidean space R_n, which subspaces do not have any common nonzero vectors, is used for the minimum

of the angles between the nonzero vectors x_1 and x_2, where $x_1 \in L_1$, $x_2 \in L_2$. If the intersection $L_1 \cap L_2 = D \neq 0$, and $D \neq L_1$, $D \neq L_2$, then the angle between L_1 and L_2 is the angle between their intersections with the orthogonal complement D^* to the intersection D. If one of the given subspaces is contained in the other (in particular, if they coincide), then the angle between them is assumed to be zero. The angle between linear manifolds is the angle between the subspaces that correspond to them. Show that the angle between any subspaces or manifolds is always defined and is zero if and only if one of the subspaces or manifolds is contained in the other or the manifolds are parallel.

*1405. Find the angle between the two-dimensional faces $A_0A_1A_2$ and $A_0A_3A_4$ of a regular four-dimensional simplex (see problem 1378) $A_0A_1A_2A_3A_4$.

*1406. Find the angle between the planes $a_0 + a_1t_1 + a_2t_2$ and $b_0 + b_1t_1 + b_2t_2$, where

$$a_0 = (3, 1, 0, 1), \quad a_1 = (1, 0, 0, 0), \quad a_2 = (0, 1, 0, 0),$$
$$b_0 = (2, 1, 1, 3), \quad b_1 = (1, 1, 1, 1), \quad b_2 = (1, -1, 1, -1).$$

*1407. Given a linearly independent set of vectors $e_1, e_2, \ldots, e_s$ and two orthogonal sets of nonzero vectors $f_1, f_2, \ldots, f_s$ and $g_1, g_2, \ldots, g_s$ such that the vectors f_k and g_k are linearly expressible in terms of $e_1, e_2, \ldots, e_k$ $(k = 1, 2, \ldots, s)$. Prove that $f_k = \alpha_k g_k$ $(k = 1, 2, \ldots, s)$, where $\alpha_k \neq 0$.

*1408. Let R_{n+1} be a Euclidean space, the vectors of which are all polynomials of degree $\leqslant n$ in the sole unknown x with real coefficients, and the scalar product of the polynomials $f(x)$ and $g(x)$ is defined thus:

$$(f, g) = \int_{-1}^{+1} f(x)\, g(x)\, dx.$$

Prove that the following polynomials (called Legendre polynomials)

$$P_0(x) = 1, \qquad P_k(x) = \frac{1}{2^k k!} \cdot \frac{d^k}{dx^k}[(x^2-1)^k] \quad (k = 1, 2, \ldots, n)$$

form an orthogonal basis of the space R_{n+1}.

1409. Using the definition of a Legendre polynomial given in the preceding problem, find the polynomials $P_k(x)$

for $k = 0, 1, 2, 3, 4$. Convince yourself that $P_k(x)$ is of degree k and write out in full the expression for $P_k(x)$ in powers of x for arbitrary k.

*1410. Compute the "length" of the Legendre polynomial $P_k(x)$ as a vector of the Euclidean space R_{n+1} of problem 1408.

*1411. Compute the value of the Legendre polynomial $P_k(x)$ for $x = 1$.

*1412. Give prove that if the orthogonalization process is applied to the basis $1, x, x^2, \ldots, x^n$ of the Euclidean space R_{n+1} of problem 1408, the result will be the polynomials $f_0(x), f_1(x), \ldots, f_n(x)$ that differ from the corresponding Legendre polynomials by constant factors only. Find those factors.

1413. Suppose the vectors $a_1, a_2, \ldots, a_n$ are carried into the vectors $b_1, b_2, \ldots, b_n$ respectively by orthogonalization. Prove that b_k is an orthogonal component of the vector a_k relative to the linear subspace L_{k-1} spanned by $a_1, \ldots, a_{k-1}$ $(k > 1)$. Furthermore, prove that

$$0 \leqslant |b_k| \leqslant |a_k| \quad (k = 1, 2, \ldots, n).$$

Here, $|b_k| = 0$ if and only if a_k is linearly expressible in terms of $a_1, \ldots, a_{k-1}$ $(k > 1)$ or $a_1 = 0$ $(k = 1)$; $|b_k| = |a_k|$ if and only if $(a_k, a_j) = 0$ $(j = 1, 2, \ldots, k-1;\ k > 1)$ or $k = 1$.

*1414. Prove that the integral $\int_{-1}^{+1} [f(x)]^2\,dx$ where $f(x)$ is an nth-degree polynomial with real coefficients and leading coefficient unity, attains its minimum, equal to $\frac{2^{2n+1}}{(2n+1)(C_{2n}^n)^2}$ if and only if $f(x) = \frac{2^n}{C_{2n}^n} P_n(x)$, where $P_n(x)$ is a Legendre polynomial of degree n (see problem 1408).

1415. The Gram determinant of the vectors $a_1, a_2, \ldots, a_k$ of the Euclidean (or unitary) space R_n is the determinant

$$g(a_1, \ldots, a_k) = \begin{vmatrix} (a_1, a_1) & (a_1, a_2) & \ldots & (a_1, a_k) \\ (a_2, a_1) & (a_2, a_2) & \ldots & (a_2, a_k) \\ \ldots & \ldots & \ldots & \ldots \\ (a_k, a_1) & (a_k, a_2) & \ldots & (a_k, a_k) \end{vmatrix}.$$

Prove that the Gram determinant does not change when orthogonalization is applied to the vectors $a_1, \ldots, a_k$; that is to say, if after orthogonalization the vectors $a_1, \ldots, a_k$ are carried into the vectors $b_1, \ldots, b_k$, then

$$g(a_1, \ldots, a_k) = g(b_1, \ldots, b_k) = (b_1, b_1)(b_2, b_2) \ldots (b_k b_k).$$

Using this fact, determine the geometric meaning of $g(a_1, a_2)$ and $g(a_1, a_2, a_3)$ on the assumption that the vectors are linearly independent.

*1416. Prove that for the vectors $a_1, \ldots, a_k$ of a Euclidean (or unitary) space to be linearly dependent, it is necessary and sufficient for the Gram determinant of these vectors to be zero.

*1417. Two bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of a Euclidean (or unitary) space are said to be reciprocal if

$$(e_i, f_j) = \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j. \end{cases}$$

Prove that for any basis $e_1, \ldots, e_n$ the reciprocal basis exists and is uniquely defined.

1418. Let S be the transition matrix from the basis $e_1, \ldots, e_n$ to the basis $e'_1, \ldots, e'_n$. Find the transition matrix T from the basis $f_1, \ldots, f_n$, that is reciprocal to the basis $e'_1, \ldots, e'_n$, to the basis $f'_1, \ldots, f'_n$, that is reciprocal to the basis $e'_1, \ldots, e'_n$,

(a) in Euclidean space, (b) in unitary space.

*1419. Prove that the Gram determinant $g(a_1, \ldots, a_k)$ is equal to zero if the vectors $a_1, \ldots, a_k$ are linearly dependent, and is positive if they are linearly independent.

1420. Prove that if the linearly independent vectors $a_1, \ldots, a_n$ are carried into the vectors $b_1, \ldots, b_n$ by orthogonalization, then $|b_k|^2 = \frac{g(a_1, \ldots, a_k)}{g(a_1, \ldots, a_{k-1})}$ $(k = 1, 2, \ldots, n)$; (the Gram determinant with a zero number of vectors is assumed equal to unity).

*1421. In the space of polynomials of degree not exceeding n in a single unknown x with real coefficients, the scalar product is given by the equation $(f, g) = \int_0^1 f(x)\, g(x)\, dx$.

Find the distance from the coordinate origin to the linear manifold consisting of all nth-degree polynomials with leading coefficient unity.

*1422. Prove that the following equation holds true for the Gram determinant:

$$0 \leqslant g(a_1, \ldots, a_k) \leqslant |a_1|^2 \ldots |a_k|^2.$$

Here, $g(a_1, \ldots, a_k) = 0$ if and only if the vectors $a_1, \ldots, a_k$ are linearly dependent, and $g(a_1, \ldots, a_k) = |a_1|^2 \ldots |a_k|^2$ if and only if the vectors $a_1, \ldots, a_k$ are pairwise orthogonal or at least one of them is zero.

1423. Use the preceding problem to prove the Hadamard inequality, namely, that if $D = |a_{ij}|_1^n$ is a determinant with real elements, then $D^2 \leqslant \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$ (see problem 923) and equality occurs if and only if either

$$\sum_{k=1}^{n} a_{ik}a_{jk} = 0 \quad (i \neq j;\ i, j = 1, 2, \ldots, n)$$

or the determinant D contains a zero row. How is the assertion altered if the determinant has complex elements?

*1424. Prove that the determinant D_f of a positive definite quadratic form $f = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ satisfies the inequality $D_f \leqslant \prod_{i=1}^{n} a_{ii}$.

*1425. Prove that any real symmetric matrix $A = (a_{ij})_1^n$ with nonnegative principal minors is a Gram matrix, that is, there exists a set of vectors $e_1, \ldots, e_n$ of the Euclidean space R_n such that $(e_i, e_j) = a_{ij}$ $(i, j = 1, 2, \ldots, n)$.

*1426. Prove that any Hermitian matrix $A = (a_{ij})_1^n$ with nonnegative principal minors is a Gram matrix, that is, there exists a set of vectors $e_1, \ldots, e_n$ of the unitary space R_n such that

$$(e_i, e_j) = a_{ij} \quad (i, j = 1, 2, \ldots, n).$$

*1427. Let us determine inductively, by the following conditions, the volume of an n-dimensional parallelepiped constructed on the linearly independent vectors $a_1, a_2, \ldots, a_n$ of a Euclidean space:

(1) $V(a_1) = |a_1|$;

(2) $V(a_1, \ldots, a_n) = V(a_1, \ldots, a_{n-1}) \cdot h_n$, where h_n is the length of the orthogonal component of the vector a_n relative to the subspace spanned by the vectors $a_1, \ldots, a_{n-1}$. Prove that

$$V(a_1, \ldots, a_n) = \sqrt{g(a_1, \ldots, a_n)} = |D|,$$

where D is a determinant of the coordinates of the given vectors in some orthonormal basis of n-dimensional space spanned by the vectors $a_1, \ldots, a_n$.

*1428. Prove that the volume of an n-dimensional parallelepiped does not exceed the product of the length of its edges emanating from a single vertex, and is equal to that product if and only if the edges are pairwise orthogonal, that is, if the parallelepiped is rectangular.

*1429. Prove the following property of a Gram determinant:

$$g(a_1, \ldots, a_k, b_1, \ldots, b_l) \leqslant g(a_1, \ldots, a_k)\, g(b_1, \ldots, b_l), \quad (1)$$

equality occurring if and only if

$$(a_i, b_j) = 0 \quad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, l) \quad (2)$$

or if at least one of the subsets $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ is linearly dependent.

*1430. Prove the following property of the volume of a parallelepiped: $V(a_1, \ldots, a_k; b_1, \ldots, b_l) \leqslant V(a_1, \ldots, a_k)\, V(b_1, \ldots, b_l)$; note that equality occurs if and only if $(a_i, b_j) = 0$ $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, l)$.

*1431. Prove that if A is a real symmetric matrix of order n with nonnegative principal minors, A_1 is a matrix of order $k < n$ in the upper left-hand corner, and A_2 is a matrix of order $n - k$ in the lower right-hand corner of A, then $|A| \leqslant |A_1| \cdot |A_2|$ (compare with problem 922).

*1432. Solve the same problem as in *1431 with the proviso that A is a Hermitian matrix with nonnegative principal minors.

*1433. Find the conditions that are necessary and sufficient for C_{n+1}^2 positive numbers

$$a_{ij} \quad (i, j = 0, 1, 2, \ldots, n;\ i > j) \quad (1)$$

to be

(a) the distances of all possible pairs of vertices of some n-dimensional simplex of the Euclidean space R_n (that is,

a set of $n + 1$ points not lying in an $(n - 1)$-dimensional linear manifold);

(b) the distances of all possible pairs of points of some set of $n + 1$ points in the Euclidean space R_n.

Sec. 18. Linear Transformations of Arbitrary Vector Spaces

In this section we are concerned mainly with linear transformations of affine vector spaces. The transformations of Euclidean and unitary spaces are considered in the next section.

Linear transformations are denoted by φ, ψ, and so on, the image of the vector x under the transformation φ by φx, the set of vectors $\varphi a_1, \ldots, \varphi a_n$ is denoted by $\varphi(a_1, \ldots, a_n)$.

The matrix of a linear transformation φ in the basis $e_1, \ldots, e_n$ is the matrix A_φ, whose columns are made up of the coordinates of the images of the basis $\varphi e_1, \ldots, \varphi e_n$ in the same basis $e_1, \ldots, e_n$; in other words, the matrix A_φ is defined by the equation

$$\varphi(e_1, \ldots, e_n) = (e_1, \ldots, e_n) A_\varphi. \tag{1}$$

Let T be the transition matrix from the basis $e_1, \ldots, e_n$ to the basis $f_1, \ldots, f_n$ (see introduction to Sec. 16), and let A_φ and B_φ be matrices of the transformation φ in the first and second bases respectively. We then have the relation

$$B_\varphi = T^{-1} A_\varphi T. \tag{2}$$

The coordinates $y_1, \ldots, y_n$ of the image φx of the vector x under a linear transformation φ are expressed in terms of the coordinates (components) $x_1, \ldots, x_n$ of the preimage x in the same basis by means of the matrix $A_\varphi = (a_{ij})_1^n$ of the linear transformation φ in the same basis in the following manner: $y_i = \sum_{j=1}^{n} a_{ij} x_j$ $(i = 1, \ldots, n)$ or, in matrix form,

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A_\varphi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{3}$$

The sum $\varphi + \psi$ and the product $\varphi\psi$ of two linear transformations φ and ψ, and the product $\alpha\varphi$ of the scalar α by the

linear transformation φ of the space R_n are transformations defined as follows:

$$(\varphi + \psi)\, x = \varphi x + \psi x, \quad (\varphi\psi)\, x = \varphi\,(\psi x), \quad (\alpha\varphi)\, x = \alpha(\varphi x)$$

for any vector x in the space R_n.

1434. Prove that the rotation of a plane through the angle α about the origin of coordinates is a linear transformation, and find the matrix of that transformation in any orthonormal basis if the positive direction for reckoning angles coincides with the smallest rotation that carries the first basis vector into the second.

1435. Prove that the rotation of three-dimensional space through an angle $\frac{2\pi}{3}$ about a straight line given in a rectangular coordinate system by the equations $x_1 = x_2 = x_3$ is a linear transformation, and find the matrix of that transformation relative to the basis of unit vectors e_1, e_2, e_3 of the coordinate axes.

1436. Prove that projecting three-dimensional space on the coordinate axis of the vector e_1 parallel to the coordinate plane of vectors e_2 and e_3 is a linear transformation; find its matrix relative to the basis e_1, e_2, e_3.

1437. Prove that projecting three-dimensional space on the coordinate plane of the vectors e_1, e_2 parallel to the coordinate axis of the vector e_3 is a linear transformation; find its matrix relative to the basis e_1, e_2, e_3.

1438. Prove that the orthogonal projecting of three-dimensional space on the axis forming equal angles with the axes of a rectangular coordinate system is a linear transformation; find its matrix relative to the basis of unit vectors of the coordinate axes.

1439. Suppose the space R_n is a direct sum of linear subspaces L_1 with basis $a_1, \ldots, a_k$ and L_2 with basis $a_{k+1}, \ldots, a_n$. Prove that the projection of the space on L_1 parallel to L_2 is a linear transformation; find the matrix of the transformation relative to the basis $a_1, \ldots, a_n$.

1440. Prove that there exists a unique linear transformation of the space R_n that carries the given linearly independent vectors $a_1, \ldots, a_n$ into the given vectors $b_1, \ldots, b_n$. The problem is to find the matrix of that transformation relative to the basis $a_1, \ldots, a_n$.

Determine which of the following transformations φ, specified by giving the coordinates of the vector φx as

functions of the coordinates of the vector x, are linear; in the case of linearity, find their matrices in the same basis in which the coordinates of the vectors x and φx are given.

1441. $\varphi x = (x_2 + x_3,\ 2x_1 + x_3,\ 3x_1 - x_2 + x_3)$.

1442. $\varphi x = (x_1,\ x_2 + 1,\ x_3 + 2)$.

1443. $\varphi x = (2x_1 + x_2,\ x_1 + x_3,\ x_3^2)$.

1444. $\varphi x = (x_1 - x_2 + x_3,\ x_3,\ x_2)$.

Prove that there exists a unique linear transformation of three-dimensional space that carries the vectors a_1, a_2, a_3 into the vectors b_1, b_2, b_3 respectively; find the matrix of the transformation relative to the same basis in which the coordinates of all vectors are given:

1445. $a_1 = (2, 3, 5)$, $b_1 = (1, 1, 1)$,
$a_2 = (0, 1, 2)$, $b_2 = (1, 1, -1)$,
$a_3 = (1, 0, 0)$; $b_3 = (2, 1, 2)$.

1446. $a_1 = (2, 0, 3)$, $b_1 = (1, 2, -1)$,
$a_2 = (4, 1, 5)$, $b_2 = (4, 5, -2)$,
$a_3 = (3, 1, 2)$; $b_3 = (1, -1, 1)$.

1447. Suppose the linear transformation φ of the space R_n carries the linearly independent vectors $a_1, \ldots, a_n$ into the vectors $b_1, \ldots, b_n$ respectively. Prove that the matrix A_φ of that transformation in some basis $e_1, \ldots, e_n$ can be found from the equation $A_\varphi = BA^{-1}$, where the columns of the matrices A and B consist of the coordinates of the vectors $a_1, \ldots, a_n$ and, respectively, $b_1, \ldots, b_n$ relative to the basis $e_1, \ldots, e_n$.

1448. Prove that the transformation of three-dimensional space $\varphi x = (x, a)\, a$, where $a = (1, 2, 3)$ is a linear transformation, and find the matrices in the orthonormal basis e_1, e_2, e_3 in which the coordinates of all vectors are given, and in the basis $b_1 = (1, 0, 1)$, $b_2 = (2, 0, -1)$, $b_3 = (1, 1, 0)$.

1449. Show that multiplication of second-order square matrices (a) on the left, and (b) on the right by a given matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, constitute linear transformations of the space of all second-order matrices; find the matrices of the

transformations relative to a basis consisting of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

1450. Demonstrate that differentiation is a linear transformation of the space of all polynomials of degree $\leqslant n$ in one unknown with real coefficients.

Find the matrix of that transformation in the basis:

(a) $1, x, x^2, \ldots, x^n$;

(b) $1, x-c, \frac{(x-c)^2}{2!}, \ldots, \frac{(x-c)^n}{n!}$, where c is a real number.

1451. What change will the matrix of a linear transformation undergo if two vectors e_i, e_j are interchanged in the basis $e_1, e_2, \ldots, e_n$?

1452. A linear transformation φ relative to the basis e_1, e_2, e_3, e_4 has the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Find the matrix of the transformation relative to the basis:

(a) e_1, e_3, e_2, e_4;

(b) $e_1, e_1 + e_2, e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_4$.

1453. The linear transformation φ has, relative to the basis e_1, e_2, e_3, the matrix

$$\begin{pmatrix} 15 & -11 & 5 \\ 20 & -15 & 8 \\ 8 & -7 & 6 \end{pmatrix}.$$

Find the matrix relative to the basis

$$f_1 = 2e_1 + 3e_2 + e_3, \quad f_2 = 3e_1 + 4e_2 + e_3,$$
$$f_3 = e_1 + 2e_2 + 2e_3.$$

1454. The linear transformation φ has, relative to the basis

$$a_1 = (8, -6, 7), \; a_2 = (-16, 7, -13), \; a_3 = (9, -3, 7),$$

the matrix

$$\begin{pmatrix} 1 & -18 & 15 \\ -1 & -22 & 20 \\ 1 & -25 & 22 \end{pmatrix}.$$

Find the matrix relative to the basis

$$b_1 = (1, -2, 1), \quad b_2 = (3, -1, 2), \quad b_3 = (2, 1, 2).$$

1455. Prove that the matrices of one and the same linear transformation in two bases coincide if and only if the transition matrix from one of the bases to the other commutes with the matrix of that linear transformation relative to one of the given bases.

1456. Prove that any linear transformation φ of one-dimensional space reduces to the multiplication of all vectors by the same scalar, that is, $\varphi x = \alpha x$ for any vector x.

1457. Let the transformation φ have the matrix $\begin{pmatrix} 3 & 5 \\ 4 & 3 \end{pmatrix}$ relative to the basis $a_1 = (1, 2)$, $a_2 = (2, 3)$. The transformation ψ has the matrix $\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$ relative to the basis $b_1 = (3, 1)$, $b_2 = (4, 2)$.

Find the matrix of the transformation $\varphi + \psi$ relative to the basis b_1, b_2.

1458. The transformation φ has the matrix $\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$ relative to the basis $a_1 = (-3, 7)$, $a_2 = (1, -2)$ and the transformation ψ has the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ relative to the basis $b_1 = (6, -7)$, $b_2 = (-5, 6)$.

Find the matrix of the transformation $\varphi\psi$ in the same basis, in which the coordinates of all vectors are given.

1459. Suppose φ is a linear transformation of the space of polynomials of degree $\leqslant n$ with real coefficients such that it carries each polynomial into its derivative. Show that $\varphi^{n+1} = 0$.

1460. Let φ be a linear transformation of differentiation, and let ψ be the multiplication by x in an infinite-dimensional space of all polynomials in x with real coefficients. Prove that $\varphi\psi^n - \psi^n\varphi = n\psi^{n-1}$.

1461. Show that the linear transformations of n-dimensional space under addition and multiplication by a scalar themselves form a vector space. Find the dimension of that space.

1462. A linear transformation φ of the space R_n is said to be *nonsingular* if its matrix A_φ in some (hence, in any) basis is nonsingular, that is, $|A_\varphi| \neq 0$. Prove that this definition is equivalent to any one of the following ones: a linear transformation φ is nonsingular if (a) from $\varphi x = 0$ follows $x = 0$; (b) under the mapping φ, any basis of space again passes into a basis; (c) the mapping φ is one-to-one, that is, if $x_1 \neq x_2$, then $\varphi x_1 \neq \varphi x_2$; (d) φ maps the space onto the whole space, that is, for any $y \in R_n$ there is an $x \in R_n$ such that $\varphi x = y$; (e) φ has an inverse transformation ψ, that is, $\psi(\varphi x) = x$ for any $x \in R_n$.

1463. Let x be an eigenvector of the linear transformation φ that belongs to the eigenvalue λ, and let $f(t)$ be a polynomial. Prove that the same vector x will be an eigenvector of the transformation $f(\varphi)$ that belongs to the eigenvalue $f(\lambda)$. In other words, prove that $\varphi x = \lambda x$ implies $f(\varphi)x = f(\lambda)x$.

***1464.** Let x be an eigenvector of the linear transformation φ belonging to the eigenvalue λ, and let $f(t)$ be a function for which the transformation $f(\varphi)$ is meaningful (if, in some basis, φ has a matrix A, then $f(\varphi)$ is defined in the same basis by the matrix $f(A)$, and it can be proved that $f(\varphi)$ does not depend on the choice of basis). Prove that the same vector x will be an eigenvector of the transformation $f(\varphi)$ which eigenvector is associated with the eigenvalue $f(\lambda)$.

Find the eigenvalues and the eigenvectors of linear transformations specified in some basis by the following matrices:

1465. $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.

1466. $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$.

1467. $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$.

1468. $\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$.

1469. $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$.

1470. $\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$.

1471. $\begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}$.

1472. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

1473. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

1474. $\begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$.

1475. Prove that the eigenvectors of a linear transformation that belong to different eigenvalues are linearly independent.

1476. Prove that any square matrix A having distinct eigenvalues is similar to a diagonal matrix (over a field containing both elements of the matrix and its eigenvalues).

1477. Prove that if the linear transformation φ of the space R_n has n distinct eigenvalues, then any linear transformation ψ that commutes with φ has a basis of eigenvectors, and any eigenvector of φ will be an eigenvector of ψ.

1478. Prove that the matrix of a linear transformation in some basis is diagonal if and only if the basis consists of the eigenvectors of the given transformation.

Determine which of the following matrices of linear transformations can be reduced to diagonal form by going over to a new basis. Find that basis and the corresponding matrix:

1479. $\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$.

1480. $\begin{pmatrix} 6 & -5 & -3 \\ 3 & -2 & -2 \\ 2 & -2 & 0 \end{pmatrix}$.

1481. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

1482. $\begin{pmatrix} 4 & -3 & 1 & 2 \\ 5 & -8 & 5 & 4 \\ 6 & -12 & 8 & 5 \\ 1 & -3 & 2 & 2 \end{pmatrix}$.

1483. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

*1484. For the matrix $A = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$

of order n, find a nonsingular matrix T for which the matrix $B = T^{-1}AT$ is diagonal, and find B.

1485. A *minimal polynomial for a vector* x relative to a linear transformation φ is a polynomial $g_x(\lambda)$ with leading coefficient 1 and having the smallest degree from among all cancelling polynomials for x relative to φ, that is, the polynomials $f(\lambda)$ with the property that $f(\varphi)x = 0$.

A similar definition is given for the minimal polynomial $g(\lambda)$ relative to a linear transformation φ for the whole space. Prove that the minimal polynomial $g(\lambda)$ of the linear transformation φ is equal to the least common multiple of the minimal polynomials for vectors of any basis of the space relative to φ.

*1486. Find the conditions for which a matrix A, which has the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ on the secondary diagonal and zeros elsewhere, is similar to a diagonal matrix.

1487. Find the eigenvalues and the eigenvectors of a linear transformation which is a differentiation of polynomials of degree $\leqslant n$ with real coefficients.

1488. Let φ be a linear transformation of the space R_n. The set of all vectors φx, where x is any vector in R_n, is termed the *image of* R_n *under the transformation* φ or the *domain of values of* φ. The collection of all vectors x in R_n such that $\varphi x = 0$ is termed the *total pre-image of zero under the transformation* φ, or *the kernel of* φ. Prove that (a) the domain of values of φ is a linear subspace of R_n whose dimension is equal to the rank of φ; (b) the kernel of φ is a linear subspace of the space R_n whose dimension is equal to the nullity of φ, that is, the difference between n and the rank of φ.

1489. Let φ be a linear transformation and let L be a subspace of R_n. Prove that (a) the image φL and (b) the total pre-image $\varphi^{-1}L$ of the subspace L under a linear transformation φ are again subspaces.

1490. Prove that for a nonsingular linear transformation φ of the space R_n the dimension of (a) the image φL and

(b) the total pre-image $\varphi^{-1}L$ of any linear subspace L is equal to the dimension of L.

*1491. Use "dim L" to denote the dimension of the linear subspace L and the notation "nullity φ" to denote the nullity of the linear transformation φ. Prove that the dimensions of the image and the total pre-image of the subspace L of the space R_n under the transformation φ satisfy the inequalities

(a) $\dim L - \text{nullity } \varphi \leqslant \dim \varphi L \leqslant \dim L$;

(b) $\dim L \leqslant \dim \varphi^{-1}L \leqslant \dim L + \text{nullity } \varphi$.

*1492. Use the preceding problem to prove the Sylveste inequality for the rank of a product of two square matrices A and B of order n: $r_A + r_B - n \leqslant r_{AB} \leqslant \min(r_A, r_B)$ (see problem 931).

1493. Prove that

(a) $\text{rank } (\varphi + \psi) \leqslant \text{rank } \varphi + \text{rank } \psi$;

(b) $\text{nullity } (\varphi\psi) \leqslant \text{nullity } \varphi + \text{nullity } \psi$ for any linear transformations φ and ψ of the space R_n.

1494. Find the eigenvalues and the eigenvectors of the linear transformation φ specified in the basis a_1, a_2, a_3, a_4 by the matrix

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}.$$

Show that the subspace spanned by the vectors $a_1 + 2a_2$ and $a_2 + a_3 + 2a_4$ is invariant under φ.

*1495. Prove that the number of linearly independent eigenvectors of the transformation φ associated with the single eigenvalue λ_0 does not exceed the multiplicity of λ_0 as a root of the characteristic polynomial of the transformation φ.

1496. Prove that a linear subspace spanned by an arbitrary set of eigenvectors of the transformation φ is invariant under φ.

1497. Prove that the set of all eigenvectors of a linear transformation φ associated with one and the same eigenvalue λ_0 (together with the zero vector) is a linear subspace that is invariant under φ.

1498. Prove that all nonzero vectors of a space are eigenvectors of the linear transformation φ if and only if φ is a similarity transformation, that is, $\varphi x = \alpha x$ with the same α for any vector x.

1499. Prove that any subspace L that is invariant under a nonsingular linear transformation φ will also be invariant under the inverse transformation φ^{-1}.

1500. Prove that (a) the image φL and (b) the total preimage $\varphi^{-1}L$ of the linear subspace L, which is invariant under the linear transformation φ, are themselves invariant under φ.

1501. Find all the linear subspaces of the space of polynomials in one unknown of degree $\leqslant n$ with real coefficients, which subspaces are invariant under the transformation φ that carries any polynomial into its derivative.

1502. Prove that the matrix of the linear transformation φ of n-dimensional space in the basis $a_1, a_2, \ldots, a_n$ is a semidisintegrated block matrix of the form

(a) $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, where A_1 is a square matrix of order $k < n$ if and only if the linear subspace spanned by the first k vectors of the basis $a_1, \ldots, a_k$ is invariant under φ;

(b) $\begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$, where A_1 is a square matrix of order $k < n$ if and only if the linear subspace spanned by the last $n - k$ vectors of the basis $a_{k+1}, \ldots, a_n$ is invariant under φ;

(c) the matrix is a disintegrated block matrix of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 is a square matrix of order k if and only if the subspace spanned by the vectors $a_1, \ldots, a_k$ and also the subspace spanned by the vectors $a_{k+1}, \ldots, a_n$ are invariant under φ.

***1503.** Let the linear transformation φ of the n-dimensional space R_n in the basis $a_1, \ldots, a_n$ have a diagonal matrix with distinct elements on the diagonal. Find all the linear subspaces that are invariant under φ and determine their number.

1504. Find all the subspaces of three-dimensional space that are invariant under a linear transformation specified

by the matrix

$$\begin{pmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

1505. Find all the subspaces of three-dimensional space that are invariant simultaneously under two linear transformations given by the matrices

$$\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} -6 & 2 & 3 \\ 2 & -3 & 6 \\ 3 & 6 & 2 \end{pmatrix}.$$

1506. Prove that any two permutable linear transformations of complex space have a common eigenvector.

1507. Prove that for any set (even infinite) of pairwise permutable linear transformations of the complex space R_n there is an eigenvector for all transformations of the given set.

1508. Prove that all root vectors of distinct eigenvalues are linearly independent.

Find the eigenvalues and the root subspaces of the linear transformations given in some basis by the following matrices:

1509. $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$.

1510. $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$.

1511. $\begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}$.

1512. $\begin{pmatrix} 0 & -2 & 3 & 2 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$.

1513. Prove that a linear transformation of complex space has a diagonal matrix in some basis if and only if all its root vectors are eigenvectors.

1514. Prove that a complex space consists solely of root vectors of a linear transformation φ if and only if all the eigenvalues of the transformation are equal.

1515. Suppose R is an infinite-dimensional space of all real functions $f(x)$ defined and having derivatives of any

order over the entire number line, with ordinary addition of functions and ordinary multiplication of a function by a scalar; and suppose φ is a transformation that carries any function into its derivative.

Find: (a) all eigenvalues and eigenvectors, (b) all root subspaces of the transformation φ.

1516. A space R_n is said to be a *cyclic* space relative to a linear transformation φ if R_n has a cyclic basis, that is, a basis $a_1, a_2, \ldots, a_n$ for which

$$\varphi a_k = a_{k+1} \quad (k = 1, 2, \ldots, n-1).$$

Prove that if R_n is a cyclic space under φ and $a_1, a_2, \ldots, a_n$ is a cyclic basis, then

(a) the minimal polynomial $g(\lambda)$ of the transformation φ is of degree n;

(b) the minimal polynomial of the whole space coincides with the minimal polynomial of the vector a_1;

(c) if $\varphi a_n = c_1 a_1 + c_2 a_2 + \ldots + c_n a_n$, then the minimal polynomial of the transformation φ is defined by

$$g(\lambda) = \lambda^n - c_n\lambda^{n-1} - c_{n-1}\lambda^{n-2} - \ldots - c_1.$$

1517. Prove that if the degree of the minimal polynomial $g(\lambda)$ of a linear transformation φ of the space R_n is n and $g(\lambda)$ is the degree of a polynomial that is irreducible over the field over which the space R_n is considered, that is, in the case of a complex space $g(\lambda) = (\lambda - \alpha)^n$, then

(a) R_n is not decomposable into a direct sum of two subspaces that are invariant under φ;

(b) R_n is cyclic under φ.

What is the form of the matrix of the transformation φ in a cyclic basis?

1518. Suppose the minimal polynomial of a linear transformation φ of the space R_n is of the form $(\lambda - \alpha)^n$. Prove that there is a vector a such that the vectors $(\varphi - \alpha\varepsilon)^{n-1} a$, $(\varphi - \alpha\varepsilon)^{n-2} a, \ldots, (\varphi - \alpha\varepsilon) a, a$, where ε is the identical transformation, form a basis of the space. What is the form of the matrix of the transformation φ in that basis?

1519. Prove that any subspace L of the complex space R_n, which subspace is invariant under the linear transformation φ, contains a straight line that is invariant under φ.

1520. Prove that any subspace L of the real space R_n, which subspace is invariant under the linear transformation φ and has an odd dimension, contains a straight line

that is invariant under φ. Show, using examples, that this assertion does not hold for a subspace of even dimension. Under what conditions does L contain a straight line, all points of which remain fixed under the transformation φ?

1521. Prove that a complex space containing only one straight line invariant under the linear transformation φ is not decomposable into a direct sum of two nonzero subspaces that are invariant under φ.

1522. Prove that under a linear transformation φ the complex space R_n decomposes into a direct sum of one or several invariant linear subspaces, each of which contains only one invariant straight line and, hence (by the preceding problem), cannot be decomposed further.

***1523.** Let φ be a linear transformation of the space R_n and let $g(\lambda)$ be the minimal polynomial of φ. Prove that

(a) if $g(\lambda) = h(\lambda)\, k(\lambda)$ and the polynomials $h(\lambda)$ and $k(\lambda)$ are coprime, then the space R_n is a direct sum of the subspaces L_1, which consists of all vectors x such that $h(\varphi)\, x = 0$, and L_2, which consists of all vectors x such that $k(\varphi)\, x = 0$;

(b) if $g(\lambda) = h_1(\lambda)\, h_2(\lambda) \ldots h_s(\lambda)$ and the polynomials $h_1(\lambda)$, $h_2(\lambda)$, ..., $h_s(\lambda)$ are pairwise coprime, then R_n is a direct sum of the subspaces L_i ($i = 1, 2, \ldots, s$), where L_i consists of all vectors x such that $h_i(\varphi)\, x = 0$.

***1524.** A linear transformation φ is given relative to the basis e_1, e_2, e_3 by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Find the minimal polynomial $g(\lambda)$ of the transformation and the decomposition of the space into a direct sum of subspaces, which decomposition corresponds to the decomposition of $g(\lambda)$ into coprime factors of the form $(\lambda - \alpha)^k$.

1525. Solve the same problem as that of 1524 if the linear transformation φ in the basis e_1, e_2, e_3 is given by the matrix

$$\begin{pmatrix} 4 & -2 & 2 \\ -5 & 7 & -5 \\ -6 & 6 & -4 \end{pmatrix}.$$

1526. A linear transformation φ of the Euclidean (or unitary) space R_n is given by $\varphi x = (x, a)\, a$ for any x in R_n; a is a given nonzero vector. Find the minimal polynomial $g(\lambda)$ of the transformation and the decomposition of the space into a direct sum, the decomposition corresponding to the decomposition of $g(\lambda)$ into coprime degrees of irreducible polynomials with real coefficients (or polynomials of the form $\lambda - \alpha$ in the case of a unitary space).

1527. Find the Jordan form of the matrix of a linear transformation φ of the complex space R_n if φ has, up to a numerical factor, only one eigenvector.

1528. Prove that the number of linearly independent eigenvectors of a linear transformation φ that belong to one and the same eigenvalue λ_0 is equal to the number of blocks with diagonal element λ_0 in the Jordan form of the matrix of φ.

***1529.** Prove that the basis in which the matrix of the linear transformation φ of the complex vector space R_n has Jordan form can be constructed in the following manner:

(A) If not all eigenvalues of φ are equal and the characteristic polynomial is of the form

$$f(\lambda) = (\lambda - \lambda_1)^{k_1} \ldots (\lambda - \lambda_s)^{k_s} \qquad (\lambda_i \neq \lambda_j \text{ for } i \neq j),$$

then we construct a basis of the subspace P_i of all vectors x such that $(\varphi - \lambda_i \varepsilon)^{k_i} x = 0$ (ε is the identical transformation) $(i = 1, 2, \ldots, s)$.

The space R_n will be a direct sum of the subspaces P_i. They are invariant under φ; φ in P_i has one eigenvalue λ_i, and $(\varphi - \lambda_i \varepsilon)^{k_i} x = 0$ for any vector x in P_i. In this construction, we can take the minimal polynomial $g(\lambda)$ instead of the characteristic polynomial $f(\lambda)$ but this may lower the exponents k_i.

(B) Let φ in R_n have a unique eigenvalue λ_0 and let k be the smallest positive integer such that $(\varphi - \lambda_0 \varepsilon)^k = 0$. Set $\psi = \varphi - \lambda_0 \varepsilon$. The *height* of the vector x is the smallest h such that $\psi^h x = 0$. We denote by R_h the subspace of all vectors of height $\leqslant h$ $(0 \leqslant h \leqslant k)$. R_0 contains only the zero vector; R_k coincides with the entire space.

We construct the basis of R_1, complete it to the basis of R_2, which in turn is extended to R_3 and so on, until the basis of R_k is completed (for brevity, we call these bases the initial bases). For each vector f of height k of the initial basis of R_k

we construct a *series* of vectors f, ψf, $\psi^2 f$, . . ., $\psi^{k-1} f$ with initial vector f. We then take any basis of R_{k-2} (say the initial basis) and vectors of height $k-1$ of all the constructed series. Taken together, these vectors are linearly independent. Now, using any vectors (say, from the initial basis of R_{k-1}) we complete them up to the basis of R_{k-1}. For each of the supplemental vectors f (if they exist) we construct a new series: f, ψf, $\psi^2 f$, . . ., $\psi^{k-2} f$, and so forth.

Suppose that, at a certain stage, certain series have been constructed in which the vectors of height $h+1$ together with any (say, the initial) basis of R_h form a basis of R_{h+1}. The vectors of any (say, the initial) basis of R_{h-1} together with vectors of height h of the constructed series will be linearly independent. We complete them to the basis of R_h using any vectors (say, from the initial basis of R_h). For each additionally taken vector f (if such exist) we construct a new series f, ψf, $\psi^2 f$, . . ., $\psi^{h-1} f$. We proceed in this fashion until the vectors of all the constructed series form a basis of the entire space. Writing down the vectors one series after another so that in each series the vectors are taken in reverse order (the initial vector of a series is last in the given series), we get the desired basis in which the matrix of the transformation φ is of Jordan form.

(C) The basis whose construction is given in (A) and (B) is not defined uniquely. Prove the uniqueness (up to the order of arrangement of the Jordan submatrices) of the Jordan matrix A_J which is similar to the given square matrix A (and, hence, the uniqueness of the Jordan form of the matrix of the given linear transformation φ). Namely, prove that the Jordan form A_J of the matrix A of order n is determined as follows. Let k be the highest order of the Jordan submatrices of the matrix A_J with the number λ_0 on the diagonal, let x_h be the number of such submatrices of order h ($h = 1, 2, \ldots, k$), $B = A - \lambda_0 E$, and let r_h be the rank of the matrix B^h ($h = 0, 1, 2, \ldots, k, k+1$). Then the numbers x_h are given by the formulas

$$x_h = r_{h-1} - 2r_h + r_{h+1} \quad (h = 1, 2, \ldots, k). \qquad (\alpha)$$

Note. The formulas (α) offer a procedure for finding the Jordan form A_J without applying to the theory of elementary divisors of λ-matrices.

The linear transformation φ of space R_n in the basis $e_1, \ldots, e_n$ is given by the matrix A. Find the basis $f_1, \ldots$

$\ldots, f_n$ in which the matrix of the transformation has the Jordan form A_J and find this Jordan form (the desired basis is not defined uniquely).

1530. $A = \begin{pmatrix} 3 & 2 & -3 \\ 4 & 10 & -12 \\ 3 & 6 & -7 \end{pmatrix}$. **1531.** $A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix}$.

1532. $A = \begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$. **1533.** $A = \begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 1 & 2 & -2 \end{pmatrix}$.

1534. $A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 2 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}$. **1535.** $A = \begin{pmatrix} 6 & -9 & 5 & 4 \\ 7 & -13 & 8 & 7 \\ 8 & -17 & 11 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix}$.

1536. $A = B^2$, where $B = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$

is a Jordan submatrix of order n.

*1537. A linear transformation φ of the space R_n is said to be an *involutory* transformation if $\varphi^2 = \varepsilon$, where ε is the identical transformation. Determine the geometrical meaning of an involutory transformation.

*1538. A linear transformation φ of the space R_n is said to be *idempotent* if $\varphi^2 = \varphi$. Determine the geometrical meaning of an idempotent transformation.

1539. Give examples of a linear transformation φ of three-dimensional space for which

(a) the space is not a direct sum of the domain of values of L_1 and of the kernel of L_2 of the transformation φ (the definition is given in problem 1488);

(b) the space is a direct sum of the domain of values of L_1 and of the kernel of L_2 for φ, but φ is not a projection on L_1 parallel to L_2.

Sec. 19. Linear Transformations of Euclidean and Unitary Vector Spaces

1540. Prove that the operation of passing from a linear transformation φ of unitary (or Euclidean) space to the conjugate transformation φ^* has the following properties:

(a) $(\varphi^*)^* = \varphi$,

(b) $(\varphi + \psi)^* = \varphi^* + \psi^*$,

(c) $(\varphi\psi)^* = \psi^*\varphi^*$,

(d) $(\alpha\varphi)^* = \bar{\alpha}\varphi^*$,

(e) if φ is nonsingular, then $(\varphi^{-1})^* = (\varphi^*)^{-1}$.

1541. Suppose e_1 and e_2 constitute an orthonormal basis of a plane and the linear transformation φ in the basis $f_1 = e_1$, $f_2 = e_1 + e_2$ has the matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. Find the matrix of the conjugate transformation φ^* in the same basis f_1, f_2.

1542. A linear transformation φ of Euclidean space in a basis made up of the vectors $f_1 = (1, 2, 1)$, $f_2 = (1, 1, 2)$, $f_3 = (1, 1, 0)$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}.$$

Find the matrix of the conjugate transformation φ^* in the same basis, assuming that the coordinates of the vectors of the basis are given in some orthonormal basis.

1543. Find the matrix of the linear transformation φ^* conjugate to the transformation φ in the orthonormal basis e_1, e_2, e_3 if φ carries the vectors $a_1 = (0, 0, 1)$, $a_2 = (0, 1, 1)$, $a_3 = (1, 1, 1)$ into the vectors $b_1 = (1, 2, 1)$, $b_2 = (3, 1, 2)$, $b_3 = (7, -1, 4)$, respectively, where the coordinates of all vectors are given in the basis e_1, e_2, e_3.

1544. Let xOy be a rectangular system of coordinates in the plane and let φ be a projection of the plane on the x-axis parallel to the bisector of the first and third quadrants. Find the conjugate transformation φ^*.

***1545.** Let $R_n = L_1 + L_2$ be a decomposition of a Euclidean (or unitary) space into a direct sum of two sub-

spaces; let φ be the projection of R_n on L_1 parallel to L_2, L_1^* and L_2^* are the orthogonal complements, respectively, of L_1 and L_2; φ^* is the conjugate transformation of φ. Prove that $R_n = L_1^* + L_2^*$ and that φ^* is a projection of R_n on L_2^* parallel to L_1^*.

1546. Prove that if the subspace L of a unitary (or Euclidean) space is invariant under a linear transformation φ, then the orthogonal complement L^* is invariant under the conjugate transformation φ^*.

***1547.** Prove that the linear transformation φ of the unitary space R_n has an invariant subspace of any number of dimensions from zero to n.

***1548.** Prove that for any linear transformation φ of unitary space there is an orthonormal basis in which the matrix of that transformation has triangular form (Schur's theorem).

1549. Write down the equation of a plane invariant under a linear transformation φ specified in some orthonormal basis by the matrix

$$\begin{pmatrix} 4 & -23 & 17 \\ 11 & -43 & 30 \\ 15 & -54 & 37 \end{pmatrix}.$$

1550. Prove that if one and the same vector x is an eigenvector for the linear transformation φ with the eigenvalue λ_1 and for the conjugate transformation φ^* with the eigenvalue λ_2, then $\lambda_1 = \overline{\lambda}_2$.

1551. Prove that if a linear transformation φ of the unitary space R_n has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of the conjugate transformation φ^* are the conjugate numbers $\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n$.

1552. Prove that the corresponding coefficients of the minimal polynomials of conjugate linear transformations are conjugate to one another.

***1553.** Let a linear transformation φ of a unitary (or Euclidean) space in the basis $e_1, \ldots, e_n$ have a matrix A, and let the conjugate transformation φ^* in the reciprocal basis (see problem 1417) $f_1, \ldots, f_n$ have the matrix B. Prove that $B = \overline{A}'$ in unitary space and $B = A'$ in Euclidean space.

***1554.** Let the scalar product (x, y) in some basis be given by the bilinear form f with matrix U (in other words

the matrix U is the Gram matrix of the vectors of the basis). Show that the matrix A of the linear transformation φ and the matrix A_1 of the conjugate transformation φ^* in the given basis are related thus:

(a) $A_1 = U^{-1}A'U$ for Euclidean space;

(b) $\bar{A}_1 = U^{-1}A'U$ for unitary space.

Suppose that in some basis the scalar product is given by the bilinear form f and the linear transformation φ is given by the matrix A. Find the matrix A_1 of the conjugate transformation φ^* in the same basis:

1555. $f = x_1y_1 + 5x_2y_2 + 6x_3y_3 + 2x_1y_3 + 2x_3y_1 + 3x_2y_3 + 3x_3y_2$;

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 0 & -1 \\ 3 & -2 & 0 \end{pmatrix}.$$

1556. $f = 2x_1y_1 + 3x_2y_2 + x_3y_3 + 2x_1y_2 + 2x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2$;

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Let U be the Gram matrix of some basis and let A be the matrix of the linear transformation φ. Find the matrix A_1 of the conjugate transformation φ^* in the same basis:

1557. $U = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$; $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$.

1558. $U = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$; $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 3 & 2 & -1 \end{pmatrix}$.

1559. Suppose φ is a linear transformation of a unitary or Euclidean space. Prove that $(e^{\varphi})^* = e^{\varphi^*}$. (The definition of a function of a linear transformation is given in problem 1464.)

1560. Prove that the product of two orthogonal (or unitary) transformations is itself orthogonal (or unitary).

1561. Prove that if a linear transformation φ of a unitary (or Euclidean) space preserves the lengths of all vectors, then it is unitary (or, respectively, orthogonal).

***1562.** Given in a unitary (or Euclidean) space some transformation φ under which each vector x is associated with a unique vector φx. Prove that if the transformation φ preserves the scalar product, that is $(\varphi x, \varphi y) = (x, y)$ for any vectors x, y of the space, then φ will be a linear and, hence unitary (or, respectively, orthogonal) transformation. Use examples to show that preservation of the scalar squares of all vectors is not sufficient for the linearity of φ.

1563. Suppose the scalar multiplication of the vectors of the space R_n is given by the Gram matrix U of the vectors of some basis. Find the condition that is necessary and sufficient for the linear transformation φ specified in the same basis by the matrix A to be

(a) an orthogonal transformation of Euclidean space;

(b) a unitary transformation of unitary space.

1564. Prove that if two vectors x and y of Euclidean (or unitary) space have the same length, then there is an orthogonal (or, respectively, unitary) transformation φ that carries x into y.

1565. Prove that if two pairs of vectors x_1, x_2 and y_1, y_2 of Euclidean (or unitary) space have the properties $|x_1| = |y_1|$, $|x_2| = |y_2|$ and the angle between x_1 and x_2 is equal to the angle between y_1 and y_2, then there is an orthogonal (respectively, unitary) transformation φ such that $\varphi x_1 = y_1$, $\varphi x_2 = y_2$.

***1566.** Given two sets of vectors $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ of Euclidean (or unitary) space. Prove the following: for an orthogonal (or, respectively, unitary) transformation φ to exist such that $\varphi x_i = y_i$ $(i = 1, 2, \ldots, k)$, it is necessary and sufficient for the Gram matrices of the sets of vectors to coincide:

$$((x_i, x_j))_1^k = ((y_i, y_j))_1^k.$$

***1567.** Let φ be a unitary (or orthogonal) transformation of the unitary (or, respectively, Euclidean) space R_n. Prove that the orthogonal complement L^* of the linear subspace L, which is invariant under φ, is also invariant under φ.

1568. Prove that two permutable unitary transformations of unitary space have a common orthonormal basis of eigenvectors.

*1569. Prove that for a unitary transformation φ of unitary space

(a) the eigenvalues are equal to unity in modulus (and, hence, the eigenvalues of the unitary and, in particular, real orthogonal, matrix are equal to unity in modulus);

(b) the eigenvectors associated with two distinct eigenvalues are orthogonal;

(c) if in a certain basis the matrix A of the transformation φ is real and the eigenvector associated with the complex eigenvalue $\alpha + \beta i$ $(\beta \neq 0)$ is represented as $x + yi$, where the vectors x and y have real coordinates (components), then x and y are orthogonal and have the same length; also note that

$$\varphi x = \alpha x - \beta y; \quad \varphi y = \beta x + \alpha y; \tag{1}$$

(d) an orthogonal transformation of Euclidean space always has a one-dimensional or two-dimensional invariant subspace.

*1570. Prove that

(a) for any unitary transformation φ of the unitary space R_n there is an orthonormal basis consisting of eigenvectors of the transformation φ. In this basis, the matrix of φ is diagonal with diagonal elements equal to unity in absolute value.

What property of unitary matrices follows from this fact?

(b) for any orthogonal transformation φ of Euclidean space R_n there is an orthonormal basis in which the matrix of φ is of canonical form, where the principal diagonal has blocks of second order of the form

$$\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \quad (\gamma \neq k\pi)$$

and blocks of first order of the form (± 1).

Blocks of any of these types may be absent. All other elements are zero. What is the geometrical meaning of the transformation? What property of real orthogonal matrices follows from this fact?

Relative to an orthogonal transformation φ given in an orthonormal basis by the matrix A, find the orthonormal basis in which the matrix B of that transformation has the canonical form indicated in problem 1570. Find that canonical form. (The desired basis is not defined uniquely.)

1571.

$$A=\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

1572.

$$A=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \end{pmatrix}.$$

1573.

$$A=\begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{2} & \frac{2}{3}\sqrt{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Find the canonical form B of the orthogonal matrix A and the orthogonal matrix Q such that $B = Q^{-1}AQ$:

1574.

$$A=\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

1575.

$$A=\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4}\sqrt{6} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4}\sqrt{6} \\ \frac{1}{4}\sqrt{6} & -\frac{1}{4}\sqrt{6} & \frac{1}{2} \end{pmatrix}.$$

1576.

$$A=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

1577.

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

1578. For a given unitary matrix

$$A = \frac{1}{9}\begin{pmatrix} 4+3i & 4i & -6-2i \\ -4i & 4-3i & -2-6i \\ 6+2i & -2-6i & 1 \end{pmatrix}$$

find the diagonal matrix B and a unitary matrix Q such that

$$B = Q^{-1}AQ.$$

1579. Prove that a linear combination of self-conjugate transformations with real coefficients (in particular, the sum of two self-conjugate transformations) is a self-conjugate transformation.

1580. Prove that the product $\varphi\psi$ of two self-conjugate transformations φ and ψ is self-conjugate if and only if φ and ψ are permutable.

1581. Prove that if φ and ψ are self-conjugate transformations, then so also will the following be self-conjugate:

$$\varphi\psi + \psi\varphi \text{ and } i(\varphi\psi - \psi\varphi).$$

1582. Prove that the reflection of φ of the Euclidean (or unitary) space R_n in the subspace L_1 parallel to the subspace L_2 is a self-conjugate linear transformation if and only if L_1 and L_2 are orthogonal.

1583. Prove that the projection φ of the Euclidean (or unitary) space R_n on the subspace L_1 parallel to the subspace L_2 is a self-conjugate linear transformation if and only if L_1 and L_2 are orthogonal.

1584. Prove that if the linear transformation φ of the unitary (or Euclidean) space R_n has any two of the following three properties:

(1) φ is a self-conjugate transformation;

(2) φ is a unitary (or, respectively, orthogonal) transformation;

(3) φ is an involutory transformation, that is, $\varphi^2 = \varepsilon$ (ε is the identical transformation), then it has the third property as well. Find all types of transformations possessing all these properties.

Find the orthonormal basis of eigenvectors and the matrix B in that basis for a linear transformation specified in some orthonormal basis by the matrix A (the desired basis is not defined uniquely):

1585. $A = \begin{pmatrix} 11 & 2 & -8 \\ 2 & 2 & 10 \\ -8 & 10 & 5 \end{pmatrix}$. **1586.** $A = \begin{pmatrix} 17 & -8 & 4 \\ -8 & 17 & -4 \\ 4 & -4 & 11 \end{pmatrix}$.

1587. $A = \begin{pmatrix} 3 & -i & 0 \\ i & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

For a given matrix A find the diagonal matrix B and the unitary matrix C such that $B = C^{-1}AC$.

1588. $A = \begin{pmatrix} 3 & 2+2i \\ 2-2i & 1 \end{pmatrix}$.

1589. $A = \begin{pmatrix} 3 & 2-i \\ 2+i & 7 \end{pmatrix}$.

***1590.** Consider an n^2-dimensional space of all complex square matrices of order n with the ordinary operations of adding matrices and multiplying a matrix by a scalar. Turn that space into a unitary space, assuming that the scalar product of two matrices $A = (a_{ij})_1^n$ and $B = (b_{ij})_1^n$ is given by the equation $(AB) = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}$.

Prove that

(a) the premultiplication of all matrices by one and the same matrix C is a linear transformation;

(b) the unitary matrices have, as vectors of the indicated space, the length $\sqrt{n}$;

(c) the premultiplication of all matrices by the conjugate transposes C and $\bar{C}'$ generates conjugate transformations;

(d) premultiplying by the unitary matrix C generates a unitary transformation;

(e) multiplying by a Hermitian matrix generates a self-conjugate transformation;

(f) multiplying by a skew-Hermitian matrix generates a skew-symmetric transformation.

1591. Suppose a scalar multiplication of vectors of the space R_n is given by the Gram matrix U of a certain basis. Find the necessary and sufficient condition for the linear transformation φ specified in the same basis by the matrix A to be self-conjugate in the case of (a) a Euclidean space, and (b) a unitary space.

***1592.** Prove that two self-conjugate transformations φ and ψ of the unitary (or Euclidean) space R_n has a common orthonormal basis of eigenvectors of both transformations if and only if the transformations commute. What property of quadratic forms and second-degree surfaces follow from this fact?

1593. Let R be a Euclidean space of dimension n^2 whose vectors are all real matrices of order n with ordinary addition of matrices and ordinary multiplication of a matrix by a scalar, and let the scalar product of the matrices $A = (a_{ij})$ and $B = (b_{ij})$ be given by $(AB) = \sum_{i,j=1}^{n} a_{ij}b_{ij}$. Furthermore, let P and Q be real symmetric matrices of order n.

Prove that the linear transformations $\varphi X = PX$ and $\psi X = XQ$ (X is any matrix of the space R) are permutable self-conjugate transformations of the space R, and determine the relationship between the common orthonormal basis of eigenvectors of the transformations φ and ψ and the orthonormal bases of the eigenvectors of the matrices P and Q.

1594. A self-conjugate linear transformation φ of the unitary (or Euclidean) space R_n is said to be positive definite if $(\varphi x, x) > 0$ and nonnegative if $(\varphi x, x) \geqslant 0$ for any vector $x \neq 0$ in R_n. Prove that the self-conjugate transformation φ is positive definite (or nonnegative) if and only if all its eigenvalues are positive (or, respectively, nonnegative). Show that for any linear transformation φ (and not only a self-conjugate transformation) $(\varphi x, x) > 0$ (or $\geqslant 0$) implies that all eigenvalues of φ are positive (or, respectively, nonnegative). Give an example showing that the converse may not hold true for a nonself-conjugate linear transformation.

*1595. Prove that if $\varphi = \psi\chi$ or $\varphi = \chi\psi$, where φ and ψ are self-conjugate linear transformations with positive eigenvalues and χ is a unitary transformation, then $\varphi = \psi$ and χ is an identical transformation (see problem 1276c).

*1596. Prove that any nonsingular linear transformation φ of unitary (or Euclidean) space can be expressed in the form $\varphi = \psi_1\chi_1$ and also in the form $\varphi = \chi_2\psi_2$, where ψ_1, ψ_2 are self-conjugate transformations with positive eigenvalues and χ_1, χ_2 are unitary (or, respectively, orthogonal) transformations; note that both the indicated representations are unique.

1597. Explain why the equalities

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are not in contradiction with the uniqueness of representation given in the preceding problem.

Represent the following matrices as products of a symmetric matrix with positive eigenvalues into an orthogonal matrix:

1598. $\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$.

1599. $\begin{pmatrix} 1 & -4 \\ 1 & 4 \end{pmatrix}$.

1600. $\begin{pmatrix} 4 & -2 & 2 \\ 4 & 4 & -1 \\ -2 & 4 & 2 \end{pmatrix}$.

1601. Prove that the self-conjugate linear transformation φ is positive definite if and only if the coefficients of its characteristic polynomial $\lambda^n + c_1\lambda^{n-1} + \ldots + c_n$ are all nonzero and have alternating signs, and is not negative (that is, has nonnegative eigenvalues) if and only if the coefficients $c_0 = 1$, c_1, c_2, $\ldots$, c_k are nonzero and have alternating signs and c_{k+1}, $\ldots$, c_n are zero. Here, k is any number from 0 to n.

*1602. Prove that if φ and ψ are self-conjugate transformations and φ is positive definite, then the eigenvalues of the transformation $\varphi\psi$ are real.

*1603. Prove that if φ and ψ are self-conjugate transformations with nonnegative eigenvalues, one of which is nonsingular, then the eigenvalues of the transformation $\varphi\psi$ are real and nonnegative.

1604. Prove that the sum of two or several nonnegative self-conjugate transformations (see problem 1594) is again a nonnegative self-conjugate transformation.

***1605.** Prove that a nonnegative self-conjugate transformation of rank r is the sum of r nonnegative self-conjugate transformations of rank 1.

***1606.** Prove that a linear transformation φ of the unitary space R_n having rank unity is a nonnegative self-conjugate transformation if and only if in any orthonormal basis its matrix can be represented as $\overline{X}'X$, where X is a row of n numbers.

***1607.** Prove that if the matrices $A = (a_{ij})_1^n$ and $B = (b_{ij})_1^n$ are Hermitian and nonnegative (that is, have nonnegative eigenvalues), then the matrix $C = (c_{ij})_1^n$, where $c_{ij} = a_{ij}b_{ij}$ $(i, j = 1, 2, \ldots, n)$ is Hermitian and nonnegative (compare with problem 1220).

1608. A linear transformation φ of the Euclidean (or unitary) space R_n is said to be skew-symmetric if $\varphi^* = -\varphi$, where φ^* is the conjugate of φ. Prove that

(a) for a linear transformation φ of Euclidean space to be skew-symmetric, it is necessary and sufficient that its matrix A in any orthonormal basis be skew-symmetric, that is, that $A' = -A$;

(b) for a linear transformation φ of unitary space to be skew-symmetric, it is necessary and sufficient for its matrix A to be skew-Hermitian in any orthonormal basis, that is, $\overline{A}' = -A$.

***1609.** Prove that the orthogonal complement L^* of the linear subspace L of a Euclidean (or unitary) space, which subspace is invariant under the skew-symmetric transformation φ, is also invariant under φ.

***1610.** Prove that for the skew-symmetric transformation φ of a unitary space:

(a) the eigenvalues are pure imaginaries (and, hence, the eigenvalues of a complex skew-Hermitian matrix, in particular a real skew-symmetric matrix, are pure imaginaries);

(b) the eigenvectors associated with two distinct eigenvalues are orthogonal;

(c) if, in an orthonormal basis, the matrix A of the transformation φ is real and the eigenvector associated with the value $\beta i \neq 0$ is expressed as $x + yi$, where the vectors x and y have real coordinates, then x and y are orthogonal

and have the same length, and

$$\varphi x = -\beta y, \quad \varphi y = \beta x; \tag{1}$$

(d) a skew-symmetric transformation of Euclidean space always has a one-dimensional or two-dimensional invariant subspace.[1]

*1611. Prove that

(a) for any skew-symmetric transformation φ of the unitary space R_n there exists an orthonormal basis consisting of eigenvectors of the transformation φ. In this basis, the matrix of φ is diagonal with purely imaginary elements on the diagonal (some of these elements may be zero). What property of complex skew-Hermitian matrices does this imply?

(b) for any skew-symmetric transformation φ of the Euclidean space R_n there exists an orthonormal basis, in which the matrix has the following canonical form: on the main diagonal are second-order blocks of the form $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, where $\beta \neq 0$, and first-order zero blocks (one of the two types of block may be absent). What is the geometrical meaning of the transformation? What property of real skew-symmetric matrices does this imply?

1612. Prove that if φ is a self-conjugate transformation of a unitary space, then the transformation $\psi = i\varphi$ is skew-symmetric. Conversely, if φ is skew-symmetric, then $\psi = i\varphi$ is a self-conjugate transformation.

1613. Prove that if φ is a self-conjugate transformation of a unitary space, then the transformation $\psi = (\varphi - i\varepsilon)^{-1}(\varphi + i\varepsilon)$, where ε is the identical transformation, exists and is unitary.

*1614. Prove that the skew-symmetric and unitary transformations of a unitary space (and, respectively, the skew-symmetric and orthogonal transformations of a Euclidean space) are related as follows: if in the equation

$$\psi = (\varepsilon - \varphi)(\varepsilon + \varphi)^{-1}, \tag{1}$$

(where ε is the identical transformation) φ is a skew-symmetric transformation, then ψ is a unitary transformation in which there is no eigenvalue of -1. Conversely, if in that same equation (1), φ is a unitary transformation devoid of an eigenvalue of -1, then ψ is a skew-symmetric trans-

formation. The equation (1) defines a reciprocal one-to-one mapping of all skew-symmetric transformations into all unitary transformations devoid of the eigenvalue -1. There is a similar relationship between skew-symmetric and orthogonal transformations of Euclidean space. What properties of matrices does this imply?

1615. Show that the equation (1) of the preceding problem defines a one-to-one correspondence, firstly, between all nonsingular skew-symmetric transformations and all unitary (or, respectively, orthogonal) transformations devoid of eigenvalues ± 1, and, secondly, between all singular skew-symmetric transformations and all unitary (orthogonal) transformations having as eigenvalues $+1$ and not having as eigenvalues -1.

1616. Prove that if φ is a skew-symmetric transformation of a unitary (or Euclidean) space, then the transformation e^{φ} is unitary (respectively, orthogonal). What property of matrices does this imply?

***1617.** Prove that the function e^{φ} generates a one-to-one mapping of all self-conjugate transformations of a unitary (or Euclidean) space onto all positive definite transformations (that is, such that are self-conjugate with positive eigenvalues).

1618. A linear transformation φ of a unitary (or Euclidean) space is said to be normal if it is permutable with its conjugate transformation φ^*. Verify that self-conjugate, skew-symmetric, and unitary (or orthogonal) transformations are normal.

1619. Prove that a normal transformation of a unitary (or Euclidean) space is self-conjugate if and only if all its eigenvalues (respectively, all the roots of its characteristic equation) are real.

1620. Prove that the normal transformation of a unitary (or Euclidean) space is unitary (or, respectively, orthogonal) if and only if all its eigenvalues (or, respectively, all the roots of its characteristic equation) are equal to unity in absolute value.

1621. Prove that the normal transformation of a unitary (or Euclidean) space is skew-symmetric if and only if all its eigenvalues (or, respectively, all the roots of its characteristic equation) are pure imaginaries.

1622. Prove that a linear transformation φ of a unitary space is normal if and only if $\varphi = \psi\chi$, where φ is a self-

conjugate transformation and χ is an unitary transformation and these transformations commute.

1623. Prove that

(a) every linear transformation φ can be uniquely represented as $\varphi = \varphi_1 + \varphi_2$, where φ_1 is a self-conjugate transformation and φ_2 is a skew-symmetric transformation;

(b) for a transformation φ to be normal, it is necessary and sufficient that the transformations φ_1 and φ_2 in the preceding representation be permutable.

1624. Prove that

(a) every linear transformation φ of a unitary space can be uniquely represented in the form $\varphi = \varphi_1 + i\varphi_2$, where φ_1 and φ_2 are self-conjugate transformations;

(b) for the transformation φ to be normal, it is necessary and sufficient for the transformations φ_1 and φ_2 in the preceding representation to be permutable.

***1625.** Prove that for any (finite or infinite) collection of pairwise permutable normal transformations of the unitary space R_n there is an orthonormal basis whose vectors are eigenvectors for all transformations of the given collection.

1626. Prove that it is possible, relative to any normal transformation φ of the unitary space R_n, to take the kth root for any natural number k in the domain of normal transformations. Find the number of distinct normal transformations ψ such that $\psi^k = \varphi$.

***1627.** Prove that if x is an eigenvector of the normal transformation φ of a unitary (or Euclidean) space, which eigenvector is associated with the eigenvalue λ, then x is an eigenvector of the conjugate transformation φ^*, which eigenvector is associated with the conjugate (respectively, very same) eigenvalue $\bar{\lambda}$.

***1628.** Prove that the eigenvectors of a normal transformation that are associated with two distinct eigenvalues are orthogonal.

***1629.** Let e be the eigenvector of a normal transformation φ. Prove that the subspace L consisting of all vectors of the space that are orthogonal to e are invariant under φ.

***1630.** Prove that to achieve normality in the linear transformation φ of a unitary space it is necessary and sufficient for each eigenvector of φ to be an eigenvector of φ^* as well.

*1631. Prove that any subspace L of the unitary space R_n, which is invariant under the normal transformation φ, has an orthonormal basis consisting of the eigenvectors of the transformation φ.

*1632. A linear transformation φ of the unitary (or Euclidean) space R_n is said to have a *normal property* if the orthogonal complement L^* of every subspace L, which is invariant under φ, is itself invariant under φ. Prove the following assertion: for a linear transformation φ of a unitary (or Euclidean) space to be normal, it is necessary and sufficient that φ have the normal property.

*1633. Prove that for normality of a linear transformation φ of a unitary (or Euclidean) space it is necessary and sufficient that every subspace that is invariant under φ be invariant under φ^* as well.

Supplement

Sec. 20. Groups

1634. Determine whether each of the following sets forms a group under the indicated operation on the elements of the set:

(1) the integers under addition;

(2) even numbers under addition;

(3) the integers that are multiples of a given natural number n under addition;

(4) the powers of a given real number a, $a \neq 0, \pm 1$ with integral exponents under multiplication;

(5) nonnegative integers under addition;

(6) odd integers under addition;

(7) the integers under subtraction;

(8) the rational numbers under addition;

(9) the rational numbers under multiplication;

(10) the nonzero rationals under multiplication;

(11) the positive rationals under multiplication;

(12) the positive rationals under division;

(13) the dyadic-rational numbers, that is, rational numbers whose denominators are powers of he number 2 with integral nonnegative exponents, under addition;

(14) all rational numbers whose denominators are equal to products of primes taken from the given set M (finite or infinite) with integral nonnegative exponents (only a finite number of which can be nonzero), under addition;

(15) the nth roots of unity (both real and complex roots), under multiplication;

(16) the roots of unity of all positive integral powers, under multiplication;

(17) nth-order matrices with real elements, under multiplication;

(18) nonsingular nth-order matrices with real elements, under multiplication;

(19) nth-order matrices with integral elements, under multiplication;

(20) nth-order matrices with integral elements and with determinant unity, under multiplication;

(21) nth-order matrices with integral elements and determinant equal to ± 1, under multiplication;

(22) nth-order matrices with real elements, under addition;

(23) substitutions of the numbers $1, 2, \ldots, n$, under multiplication;

(24) even substitutions of the numbers $1, 2, \ldots, n$, under multiplication;

(25) odd substitutions of the numbers $1, 2, \ldots, n$, under multiplication;

(26) one-to-one mappings of the set $N = \{1, 2, 3, \ldots\}$ of positive integers onto itself, each mapping shifting only a finite number of numbers if for the product of mappings s and t we assume the mapping st, which is obtained when the mappings s and t are carried out in succession;

(27) transformations of the set M, that is, the one-to-one mappings of that set onto itself, if for the product of the transformations s and t we take the transformation st, which is obtained when the transformations s and t are carried out in succession;

(28) vectors of the n-dimensional linear space R_n, under addition;

(29) parallel translations of three-dimensional space R if for the product of the translations s and t we take their translations in succession;

(30) rotations of three-dimensional space R about a given point O if for the product of the rotations s and t we take their successive execution;

(31) all motions of three-dimensional space R if for the product of the motions s and t we take the motion st that results in a successive execution of the motions s and t;

(32) the positive real numbers if the operation is defined as $a * b = a^b$;

(33) the positive real numbers if the operation is defined thus: $a * b = a^2 b^2$;

(34) real polynomials of degree $\leqslant n$ (including zero) in the unknown x, under addition;

(35) real polynomials of degree n in the unknown x, under addition;

(36) real polynomials of any degrees (including zero) in the unknown x, under addition.

1635. Prove that a finite set G in which an associative algebraic operation is defined and each of the equations $ax = b$ and $ya = b$ for arbitrary a and b in G has, in G, at most one solution is a group.

1636. Prove that if $a^2 = e$ for any element a of the group G, then it is an Abelian group.

***1637.** Prove that the group of nth roots of unity is a unique multiplicative group of the nth order with numerical elements.

***1638.** Find all groups (up to within an isomorphism) of order (a) 3, (b) 4, (c) 6. Write a multiplication table of these groups and represent them in the form of substitution groups.

***1639.** Show that the rotations of each of five regular polyhedrons about the centre, which rotations bring to coincidence the polyhedron with itself, form a group if for the multiplication (composition) of two rotations we take the rotations in succession. Find the orders of the groups.

1640. Prove that the groups (1) to (4) in problem 1634 are isomorphic.

1641. Prove that

(a) all infinite cyclic groups are isomorphic among themselves;

(b) all finite cyclic groups of a given order n are isomorphic among themselves.

1642. Prove that

(a) the group of positive real numbers under multiplication is isomorphic with the group of all real numbers under addition;

(b) the group of positive rational numbers under multiplication is not isomorphic with the group of all rational numbers under addition.

***1643.** Prove that

(a) any finite group of order n is isomorphic with a certain substitution group of n elements;

(b) any group is isomorphic with the group of certain one-to-one mappings of the set of elements of the group onto itself.

1644. Prove that for any elements a, b, c of a group G

(a) the elements ab and ba are of the same order;

(b) the elements abc, bca and cab are of the same order.

1645. Prove that if e is unity and a is an element of order n of a group G, then $a^k = e$ if and only if k is divisible by n.

1646. Find all the generating elements of the additive group of integers.

1647. Let $G = \{a\}$ be a cyclic group of order n and $b = a^k$. Prove that

(a) the element b is a generator of the group G if and only if the numbers n and k are coprime;

(b) the order of the element b is equal to $\frac{n}{d}$, where d is the greatest common divisor of n and k;

(c) if n and k are coprime, then there is a root $\sqrt[k]{a}$ in G, that is, a is the kth power of some element of G and conversely;

(d) all elements are squares in a group of odd order.

***1648.** Prove the following assertions:

(a) if the elements a and b of a group G are permutable, that is,

$$ab = ba \tag{1}$$

and have finite coprime orders r and s, then their product ab is of order rs;

(b) if the elements a and b of a group G are permutable, have finite orders r and s and the intersection of their cyclic subgroups contain the unit element e alone, that is

$$\{a\} \cap \{b\} = \{e\}, \tag{2}$$

then the order of the product ab is equal to the least common multiple of r and s. Use examples to show that for this assertion to hold, the conditions (1) and (2) separately, are not sufficient and the condition (1) is not a consequence of condition (2) even for coprime orders of the elements a and b;

(c) if the orders r and s of the elements a and b are coprime, then condition (2) holds;

(d) use an example to show that without condition (2) the order of the product ab cannot be determined uniquely by the orders of the factors a and b.

1649. Which groups of problem 1634 are subgroups of the other groups?

1650. Show that

(a) if H is a finite set of elements of a group G and the product of any two elements of H again lies in H, then H is a subgroup of G;

(b) if all the elements of the set H of a group G have finite orders and the product of any two elements of H lies in H, then H is a subgroup of G.

1651. Prove that in any substitution group containing at least one odd substitution

(a) the number of even substitutions is equal to the number of odd ones,

(b) the even substitutions form a normal divisor,

(c) all simple substitution groups of n elements (where $n > 2$) are contained in the alternating group A_n (a *simple* group is one that does not contain any normal divisors except itself and the unit subgroup).

1652. Prove that any infinite group has an infinity of subgroups.

1653. Up to within an isomorphism, find all groups each of which is isomorphic to any one of its nonunit subgroups.

1654. Find all the subgroups:

(a) of the cyclic group of order 6;

(b) of the cyclic group of order 24;

(c) of the fourth group (problem 1638);

(d) of the symmetric group S_3.

(e) Which of the subgroups of the group S_3 are normal divisors?

(f) Prove that the alternating group of fourth degree A_4 has no subgroup of order six. Thus, the group G of order n may not have any subgroups of order k for certain k that divide n.

1655. Find all subgroups of the group G of order 8, all elements of which, except unity e, are of order 2.

1656. Let $G = \{a\}$ be a finite cyclic group of order n. Prove the following assertions:

(a) the order of any subgroup of group G divides the order n of that group;

(b) for any divisor d of the number n there is a unique subgroup H of the group G having order d;

(c) a subgroup H of order d contains, as generators, all elements of order d of the group G, in particular, $H = \left\{a^{\frac{n}{d}}\right\}$.

*1657. Find all subgroups of the primary cyclic group, that is, of the cyclic group $G = \{a\}$ of order p^k, where p is prime.

*1658. Prove the following statements:

(a) a symmetric group S_n for $n > 1$ is generated by the set of all transpositions (i, j);

(b) the symmetric group S_n for $n > 1$ is generated by the transpositions: $(1, 2)$, $(1, 3)$, . . ., $(1, n)$;

(c) the alternating group A_n for $n > 2$ is generated by the set of all triple cycles $(i\ j\ k)$;

(d) the alternating group A_n for $n > 2$ is generated by the triple cycles: $(1\ 2\ 3)$, $(1\ 2\ 4)$, . . ., $(1\ 2\ n)$.

1659. Find the cosets:

(a) of the additive group of integers with respect to the subgroup of numbers that are multiples of a given natural number n;

(b) of the additive group of real numbers with respect to the subgroup of integers;

(c) of the additive group of complex numbers with respect to the subgroup of Gaussian integers, that is, the numbers $a + bi$, where a and b are integers;

(d) of the additive group of vectors in the plane (vectors issuing from the coordinate origin) with respect to the subgroup of vectors lying on the axis of abscissas Ox;

(e) of the multiplicative group of complex numbers different from zero with respect to the subgroup of numbers equal to unity in absolute value;

(f) of the multiplicative group of complex numbers different from zero with respect to the subgroup of positive real numbers;

(g) of the multiplicative group of complex numbers different from zero with respect to the subgroup of real numbers;

(h) of the symmetric group S_n with respect to the subgroup of substitutions that leave the number n fixed.

*1660. Prove that

(a) the subgroup H of order k of the finite group G of order $2k$ contains the squares of all elements of G;

(b) the subgroup H of index 2 of any group G contains the squares of all elements of G.

*1661. Prove that for $n > 1$ the alternating group A_n is the sole subgroup of index two (that is, such that contains half of all the elements) in the symmetric group S_n.

Give an example of a finite group that contains several subgroups of index two.

*1662. Prove that

(a) the group of a tetrahedron is isomorphic to the group of even substitutions of four elements;

(b) the groups of a cube and an octahedron are isomorphic to the group of all substitutions of four elements;

(c) the groups of the dodecahedron and icosahedron are isomorphic to the group of even substitutions of five elements. For the definition of polyhedron groups see problem 1639.

1663. Prove that any subgroup of index two is a normal divisor.

1664. Prove that the set Z of all elements of a group G, each of which is permutable with all elements of that group, is a normal divisor (the centre of the group G).

1665. The element $aba^{-1}b^{-1}$ is termed a *commutator of the elements* a and b of a group G. Prove that all commutators and their products (with any finite number of factors) form a normal divisor K of the group G (the commutant of the given group).

1666. Prove that in the group of all motions of three-dimensional space the element $x^{-1}ax$, that is conjugate to the rotation a about a point P, is a rotation about that point Q into which P passes under the motion x.

1667. Prove that the substitution $x^{-1}ax$, which is conjugate to the substitution a in the substitution group, is obtained by applying a transforming substitution x to all numbers in the expansion of the substitution a into independent cycles.

*1668. Prove that

(a) the four-group V (problem 1638) is a normal divisor of the symmetric group S_4;

(b) the factor group S_4/V is isomorphic to the symmetric group S_3.

*1669. Using problem 1667, find the number of substitutions of the symmetric group S_n that are permutable with the given substitution s.

*1670. Prove that if the intersection of two normal divisors H_1 and H_2 of a group G contains only the unit e, then any element $h_1 \in H_1$ is permutable with any element $h_2 \in H_2$.

1671. Prove that

(a) the elements of a group G that are permutable with the given element a form a subgroup $N(a)$ of the group G (the normalizer a in G) that contains the cyclic subgroup $\{a\}$ as a normal divisor;

(b) the number of elements of the group G that are conjugate to a is equal to the index of the normalizer $N(a)$ in G.

1672. Prove that

(a) the elements of the group G that are permutable with the given subgroup H (but not necessarily with the elements of H) form a subgroup $N(H)$ of the group G (the normalizer of the subgroup H in G) that contains the subgroup H as a normal divisor;

(b) the number of subgroups of the group G that are conjugate to H is equal to the index of the normalizer $N(H)$ in G.

***1673.** Prove that

(a) the number of elements in the group G that are conjugate to the given element divides the order of G;

(b) the number of subgroups of the group G that are conjugate to the given subgroup divides the order of the group G.

1674. Use the problems 1669 and 1671 to find the number of substitutions of the symmetric group S_n that are conjugate to the given substitution s.

***1675.** Prove that the centre of a group G of order p^n, where p is prime, contains more than one element.

***1676.** Prove that any normal divisor H of the alternating group A_n of degree $n \geqslant 5$, which divisor contains at least one triple cycle, coincides with A_n.

***1677.** (a) Find all classes of conjugate elements of the icosahedron group (problem 1639); (b) prove that the icosahedron group is simple (that is, that it does not contain any normal divisor different from the group itself and from the unit subgroup).

***1678.** Prove that an alternating group of the fifth degree is simple.

1679. Prove that the group G' is a homomorphic image of a finite cyclic group G if and only if G' is also cyclic and its order divides the order of the group G.

1680. Prove that if a group G is homomorphically mapped onto the group G', an element a in G being mapped onto a' in G', then

(a) the order of a is divisible by the order of a';

(b) the order of G is divisible by the order of G'.

1681. Find all the homomorphic mappings:

(a) of a cyclic group $\{a\}$ of order n into itself;

(b) of a cyclic group $\{a\}$ of order 6 into a cyclic group $\{b\}$ of order 18;

(c) of a cyclic group $\{a\}$ of order 18 into a cyclic group $\{b\}$ of order 6;

(d) of a cyclic group $\{a\}$ of order 12 into a cyclic group $\{b\}$ of order 15;

(e) of a cyclic group $\{a\}$ of order 6 into a cyclic group $\{b\}$ of order 25.

1682. Prove that the additive group of rational numbers cannot be homomorphically mapped onto the additive group of integers.

1683. An isomorphic mapping of a group G onto itself is termed an *automorphism*, and a homomorphic mapping into itself is termed an *endomorphism* of that group. An automorphism φ is said to be *internal* if there is an element x in G such that $a\varphi = x^{-1}ax$ for any a in G, and it is said to be *external* otherwise. All automorphisms of G themselves form a group if a product of automorphisms is defined as the automorphisms taken in succession: $a(\varphi\psi) = (a\varphi)\psi$. All endomorphisms of an Abelian group G form a ring if the addition of endomorphisms is defined by the equality $a(\varphi + \psi) = a\varphi + a\psi$, and multiplication is defined in the same way as for automorphisms. Find the group of automorphisms of a cyclic group $\{a\}$ of order (a) 5 and (b) 6.

(c) Prove that the symmetric group S_3 has six internal automorphisms and not a single external one, and the group of automorphisms is isomorphic to S_3.

(d) The four-group V (problem 1638) has one internal automorphism (identical) and five external ones, and the group of automorphisms is isomorphic to S_3.

Find the ring of endomorphisms of the cyclic group $\{a\}$ of order (e) 5, (f) 6, (g) n.

1684. Prove that the factor group of a symmetric group S_n with respect to an alternating group A_n is isomorphic to the factor group of the additive group of integers with respect to the subgroup of even numbers.

1685. Find the factor groups:

(a) of the additive group of integers with respect to the subgroup of multiples of the given natural number n;

(b) of the additive group of integers that are multiples of 3 with respect to the subgroup of multiples of 15;

(c) of the additive group of integers that are multiples of 4 with respect to the subgroup of multiples of 24;

(d) of the multiplicative group of real numbers different from zero with respect to the subgroup of positive numbers.

1686. Let G_n be the additive group of vectors of an n-dimensional linear space and let H_k be the subgroup of vectors of a k-dimensional subspace, $0 \leqslant k \leqslant n$. Prove that the factor group G_n/H_k is isomorphic to G_{n-k}.

1687. Let G be the multiplicative group of all complex numbers different from zero and let H be the set of all numbers in G lying on the real and imaginary axes.

(a) Prove that H is a subgroup of the group G.

(b) Find the cosets of the group G with respect to the subgroup H.

(c) Prove that the factor group G/H is isomorphic to the multiplicative group U of all complex numbers equal to unity in absolute value.

***1688.** Let G be the multiplicative group of complex numbers different from zero and let H be the set of numbers in G that lie on n rays issuing from zero at equal angles, one of the rays coinciding with the positive real axis; let K be the additive group of all real numbers, Z be the additive group of integers, D the multiplicative group of positive numbers, U the multiplicative group of complex numbers equal to unity in absolute value, and U_n the multiplicative group of the nth roots of unity. Prove that

(a) K/Z is isomorphic to U;

(b) G/D is isomorphic to U;

(c) G/U is isomorphic to D;

(d) U/U_n is isomorphic to U;

(e) G/U_n is isomorphic to G;

(f) H is a subgroup of the group G and G/H is isomorphic to U;

(g) H/D is isomorphic to U_n;

(h) H/U_n is isomorphic to D.

1689. For multiplicative groups of nonsingular square matrices of order n, prove the following statements:

(a) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant unity is isomorphic to the multiplicative group of real numbers different from zero;

(b) the factor group of the group of real matrices with respect to the subgroup of matrices with determinant equal

to ± 1 is isomorphic to the multiplicative group of positive numbers;

(c) the factor group of the group of real matrices with respect to the subgroup of matrices with positive determinants is a cyclic group of the second order;

(d) the factor group of the group of complex matrices with respect to the subgroup of matrices with determinants equal to unity in absolute value is isomorphic to the multiplicative group of positive numbers;

(e) the factor group of the group of complex matrices with respect to the subgroup of matrices with positive determinants is isomorphic to the multiplicative group of complex numbers equal to unity in absolute value.

1690. Let G be the group of all motions of three-dimensional space, H the subgroup of parallel translations, and K the subgroup of rotations about a given point O. Prove that

(a) H is the normal divisor of the group G and K is not;

(b) the factor group G/H is isomorphic to K.

1691. Prove that the normal divisor H of the group G having the finite index j contains all elements of the group G, the orders of which are prime to j. Show, using an example, that the statement may not be true for the subgroup H that is not a normal divisor.

1692. Prove that the factor group G/H is commutative if and only if H contains the commutant K of the group G (problem 1665).

***1693.** Prove that the factor group of a noncommutative group G with respect to its centre Z (problem 1664) cannot be cyclic.

***1694.** Prove that if the order of a finite group G is divisible by the prime number p, then G contains an element of order p (Cauchy's theorem).

***1695.** Let p be a prime. The group G is called a *p-group* (in the commutative case, a *primary group*) if the orders of all its elements are finite and are equal to certain powers of the number p. Prove that the finite group G is a p-group if and only if its order is equal to the power of the number p.

1696. Prove that

(a) the additive group of vectors of n-dimensional linear space is a direct sum of n subgroups of the vectors of one-dimensional subspaces spanned by the vectors of any basis of space;

(b) the additive group of complex numbers is a direct sum of the subgroups of real numbers and pure imaginaries;

(c) the multiplicative group of real numbers is a direct product of the subgroup of positive numbers and the subgroup of numbers ± 1;

(d) the multiplicative group of complex numbers is the direct product of the subgroups of positive numbers and of numbers equal to unity in absolute value.

1697. Prove that if $G = A + B_1 = A + B_2$ are direct decompositions of the Abelian group G and if B_1 contains B_2, then $B_1 = B_2$.

1698. Prove that the subgroup H of the Abelian group G is a summand in the direct decomposition $G = H + K$ if and only if there is a homomorphic mapping of G onto H that holds all elements of H fixed.

1699. Prove that if $G = A + B$ is a direct decomposition of the group G, then the factor group G/A is isomorphic to B.

1700. Let $G = A_1 + A_2 + \ldots + A_s$ be a decomposition of the Abelian group G into a direct sum of the subgroups and let

$$x = a_1 + a_2 + \ldots + a_s, \quad a_i \in A_i, \quad i = 1, 2, \ldots, s$$

be the appropriate decomposition of the element x into a sum of components. Prove that

(a) the group G is of finite order n if and only if each subgroup A_i has finite order n_i, $i = 1, 2, \ldots, s$, and $n = n_1, n_2, \ldots, n_s$;

(b) the element x is of finite order p if and only if each of its components a_i is of finite order p_i, $i = 1, 2, \ldots, s$, and p is equal to the lowest common multiple of the numbers $p_1, p_2, \ldots, p_s$;

(c) the group G is a finite cyclic group if and only if all the direct summands A_i are finite cyclic groups and their orders are relatively prime in pairs.

1701. Decompose into a direct sum of primary cyclic subgroups the cyclic group $\{a\}$ of order (a) 6, (b) 12, (c) 60, (d) 900.

***1702.** Prove the nondecomposibility into a direct sum of two nonzero subgroups:

(a) of the additive group of integers;

(b) of the additive group of rational numbers;

(c) of the primary cyclic group.

*1703. Let G be a nonzero finite Abelian group (with additive notation for the group operation). Prove the following statements:

(a) if the orders of all elements of G divide the product pq of coprime numbers p and q, then G can be decomposed into a direct sum of subgroups A and B, where the orders of all elements of A divide p, and those of B divide q; note that one of the subgroups, A or B, may prove to be zero;

(b) for the group G we have the decomposition $G = A_1 + A_2 + \ldots + A_s$ into a direct sum of (nonzero) primary subgroups that refer appropriately to distinct primes $p_1, p_2, \ldots, p_s$. These subgroups A_i are called primary components of the group G;

(c) the primary component A_i which refers to the prime number p_i consists of all elements of the group G whose orders are equal to the powers of the number p_i, which fact uniquely determines the decomposition of the group G into primary components;

(d) the decomposition, into primary components, of the nonzero subgroup H of the group G is of the form $H = B_1 + B_2 + \ldots + B_s$, where $B_i = H \cap A_i$, $i = 1, 2, \ldots, s$; the zero subgroups B_i in the decomposition of H are omitted.

1704. Denote by $G(n_1, n_2, \ldots, n_s)$ the direct sum of the cyclic groups of orders $n_1, n_2, \ldots, n_s$ respectively. From the theory of finite Abelian groups we know that each such group can be uniquely (up to an isomorphism) represented in the form $G(n_1, n_2, \ldots, n_s)$, where the numbers n_i are equal to the powers of prime numbers (not necessarily distinct). Apply this notation to find all Abelian groups of the following orders: (a) 3, (b) 4, (c) 6, (d) 8, (e) 9, (f) 12, (g) 16, (h) 24, (i) 30, (j) 36, (k) 48, (l) 60, (m) 63, (n) 72, (o) 100.

1705. Decompose, into a direct sum of primary cyclic and infinite cyclic subgroups, the factor group G/H, where G is a free Abelian group with basis x_1, x_2, x_3 and H is a subgroup with the generators:

(a) $y_1 = 7x_1 + 2x_2 + 3x_3$,
$y_2 = 21x_1 + 8x_2 + 9x_3$,
$y_3 = 5x_1 - 4x_2 + 3x_3$;

(b) $y_1 = 4x_1 + 5x_2 + 3x_3$,
$y_2 = 5x_1 + 6x_2 + 5x_3$,
$y_3 = 8x_1 + 7x_2 + 9x_3$;

(c) $y_1 = 5x_1 + 5x_2 + 2x_3$, (d) $y_1 = 6x_1 + 5x_2 + 7x_3$,
$y_2 = 11x_1 + 8x_2 + 5x_3$, $y_2 = 8x_1 + 7x_2 + 11x_3$,
$y_3 = 17x_1 + 5x_2 + 8x_3$; $y_3 = 6x_1 + 5x_2 + 11x_3$,

(e) $y_1 = 4x_1 + 5x_2 + x_3$, (f) $y_1 = 2x_1 + 6x_2 - 2x_3$,
$y_2 = 8x_1 + 9x_2 + x_3$, $y_2 = 2x_1 + 8x_2 - 4x_3$,
$y_3 = 4x_1 + 6x_2 + 2x_3$; $y_3 = 4x_1 + 12x_2 - 4x_3$;

(g) $y_1 = 6x_1 + 5x_2 + 4x_3$, (h) $y_1 = x_1 + 2x_2 + 3x_3$,
$y_2 = 7x_1 + 6x_2 + 9x_3$, $y_2 = 2y_1$,
$y_3 = 5x_1 + 4x_2 - 4x_3$; $y_3 = 3y_1$;

(i) $y_1 = 4x_1 + 7x_2 + 3x_3$, (j) $y_1 = 2x_1 + 3x_2 + 4x_3$,
$y_2 = 2x_1 + 3x_2 + 2x_3$, $y_2 = 5x_1 + 5x_2 + 6x_3$,
$y_3 = 6x_1 + 10x_2 + 5x_3$; $y_3 = 2x_1 + 6x_2 + 9x_3$.

***1706.** Prove that the finite Abelian group G is cyclic if its order is equal to:

(a) the product of two distinct primes p and q;

(b) the product of distinct primes $p_1, p_2, \ldots, p_s$.

(c) Find all subgroups of the Abelian group G whose order satisfies (b) and find the number of such subgroups.

(d) Prove that for any divisor k of order n of a finite Abelian group G there exist a subgroup and a factor group of the group G having the order k.

***1707.** Let G be a nonzero finite Abelian group, all nonzero elements of which have one and the same order p (the elementary group). Prove the following statements:

(a) the number p is prime;

(b) the group G can be decomposed into a direct sum of a finite number of cyclic subgroups of order p and has order p^k, where k is the number of those terms;

(c) any nonzero subgroup H of the group G is itself elementary and is a direct summand in some direct decomposition $G = H + K$ of the group G;

(d) the number of subgroups of order p^l of the elementary group G of order p^k, where $k \geqslant l > 0$, is equal to
$$\frac{(p^k-1)(p^k-p)(p^k-p^2)\ldots(p^k-p^{l-1})}{(p^l-1)(p^l-p)(p^l-p^2)\ldots(p^l-p^{l-1})}.$$

***1708.** Prove that a finite Abelian group G is generated by its elements of maximum order.

Sec. 21. Rings and Fields

Determine which of the following sets are rings (but not fields) and which are fields with respect to the indicated operations. (If the operations are not indicated, then we assume addition and multiplication of the numbers.)

1709. The integers.

1710. The even numbers.

1711. Integers that are multiples of n (consider, for instance, the case $n = 0$).

1712. The rational numbers.

1713. The real numbers.

1714. The complex numbers.

1715. Numbers of the form $a + b\sqrt{2}$, where a and b are integers.

1716. Numbers of the form $a + b\sqrt{3}$, where a and b are rational.

1717. Complex numbers of the form $a + bi$, where a and b are integers.

1718. Complex numbers of the form $a + bi$, where a and b are rational.

1719. Matrices of order n with integral elements, under addition and multiplication of matrices.

1720. Matrices of order n with real elements, under addition and multiplication of matrices.

1721. Functions with real values that are continuous on the interval $[-1, +1]$, under ordinary addition and multiplication of functions.

1722. Polynomials in the single unknown x with integral coefficients, under the ordinary operations of addition and multiplication.

1723. Polynomials in one unknown x with real coefficients, under the ordinary operations of addition and multiplication.

1724. All matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ with rational or real a and b, under the ordinary operations of addition and multiplication of matrices.

***1725.** Does the set of all trigonometric polynomials $a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ with real coefficients form

a ring? Determine whether the same is true for polynomials in cosines alone: $a_0 + \sum_{k=1}^{n} a_k \cos kx$, and sines alone: $\sum_{k=1}^{n} b_k \sin kx$.

*1726. Is a ring formed by numbers of the form $a + b\sqrt[3]{2}$, where a and b are rational, under the ordinary operations (for the sake of definiteness, take the real value of the root)?

*1727. Show that numbers of the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$, where a, b and c are rational, form a field, and show that each element of the field in the indicated form can be represented uniquely. Find the inverse of the number $1 - \sqrt[3]{2} + 2\sqrt[3]{4}$ (we assume the real value of the root).

1728. Prove that numbers of the form $a + b\sqrt[3]{5} + c\sqrt[3]{25}$ with rational a, b and c form a field; find the inverse of $x = 2 + 3\sqrt[3]{5} - \sqrt[3]{25}$ in that field.

*1729. Let α be a root of the polynomial $f(x)$ of degree $n > 1$ with rational coefficients, the polynomial not being reducible over the field of rationals. Prove that numbers of the form $a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_{n-1}\alpha^{n-1}$ with rational $a_0, a_1, a_2, \ldots, a_{n-1}$ form a field, each element of the field capable of being written down uniquely in the indicated form. We say that this field is obtained by adjoining the number α to the field of rational numbers.

*1730. In the field obtained by adjoining the root α of the polynomial $f(x) = x^3 + 4x^2 + 2x - 6$ (problem 1729) to the field of rational numbers, find the inverse of $\beta = 3 - \alpha + \alpha^2$.

1731. Prove that all diagonal matrices, that is, matrices of the form

$$\begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix},$$

of order $n \geqslant 2$ with real elements form a commutative ring with zero divisors under the ordinary operations of addition and multiplication of matrices.

1732. Give examples of zero divisors in the ring of functions continuous on the interval $[-1, +1]$.

1733. Prove that in the ring of square matrices of order n with elements taken from some field, nonsingular matrices, and they alone, are divisors of zero.

1734. Show that the pairs of integers (a, b) with the operations given by the equalities

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2),$$
$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$$

form a ring, and find all zero divisors of that ring.

1735. Prove that a field does not have divisors of zero.

1736. Prove that the equality $ax = ay$ for a given element a and for any elements x and y of a ring implies the equality $x = y$ if and only if a is not a left divisor of zero.

1737. Show that matrices of order $n \geqslant 2$ with elements taken from a certain field, in which matrices all rows from the second onwards consist of zeros, form a ring, and that in the ring any nonzero element will be a right divisor of zero. What matrices in the ring are not left divisors of zero?

***1738.** Show that in a ring with unity e the commutativity of addition follows from the other axioms of the ring.

1739. Having verified that the property of zero and zero divisors can be proved without making use of the commutativity of addition, prove that in a ring containing at least one element c that is not a zero divisor, the commutativity of addition follows from the other axioms of the ring.

1740. Give examples of rings of matrices of a special kind that have several right or several left units.

1741. Given an integer $n \geqslant 0$. Two integers a and b are said to be *congruent modulo n* [in symbols: $a \equiv b \pmod{n}$], if their difference $a - b$ is divisible by n (for $n = 0$ this means that $a = b$; for $n > 0$, this means that when a and b are divided by n, they yield the same remainder called the residue modulo n). Show that the set of all integers Z can be split into classes of congruent numbers that do not have any elements in common. Let us define the addition and multiplication of classes in terms of the appropriate operations on their representatives, that is, if the numbers a, b, $a + b$ and ab belong, respectively, to the classes A, B, C and D, then we set $A + B = C$ and $AB = D$.

Prove that under such operations the set of classes is a ring (the ring of residues Z_n modulo n).

*1742. Prove that a finite commutative ring without zero divisors and containing only one element is a field.

*1743. Show that the ring of residues modulo n (problem 1741) is a field if and only if n is prime.

1744. A square matrix is said to be a *scalar matrix* if its elements on the main diagonal are equal and the elements off the diagonal are zero. Show that scalar matrices of order n with real elements under ordinary operations form a field that is isomorphic to the field of real numbers.

1745. Show that matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are real numbers, form a field that is isomorphic to the field of complex numbers.

1746. Prove that the field of matrices of the form $\begin{pmatrix} b & a \\ 2b & a \end{pmatrix}$ with rational a and b (problem 1724) is isomorphic to the field of numbers of the form $a + b\sqrt{2}$ also with rational a and b.

*1747. Prove that the algebra of real matrices of the form

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

is isomorphic to the algebra of quaternions $a + bi + cj + dk$.

1748. Prove that the algebra of matrices of the form $\begin{pmatrix} a+bi & c+di \\ +c+di & a-bi \end{pmatrix}$ with real a, b, c, d and $i = \sqrt{-1}$ is isomorphic to the algebra of quaternions $a + bi + cj + + dk$.

1749. Find all automorphisms (that is, isomorphic mappings onto themselves) of the field of complex numbers, which automorphisms leave the real numbers unchanged.

*1750. Prove that any number field contains the field of rational numbers as a subfield.

*1751. Prove that under any isomorphism of number fields the subfield of rational numbers is mapped identically.

In particular, the field of rational numbers admits only an identical isomorphic mapping into itself.

*1752. Prove that an identical mapping is the sole isomorphic mapping of the field of real numbers into itself.

1753. Use problem 1752 to find all isomorphic mappings of the field of complex numbers into itself, which mappings carry the real numbers into real numbers.

1754. Prove that the minimal subfield of any field of characteristic zero is isomorphic to the field of rational numbers.

1755. Prove that the minimal subfield of any field with characteristic p is isomorphic to the field of residues modulo p.

1756. Solve the system of equations

$$x + 2z = 1, \quad y + 2z = 2, \quad 2x + z = 1$$

in the field of residues modulo 3 and modulo 5.

1757. Solve the system of equations

$$3x + y + 2z = 1, \quad x + 2y + 3z = 1, \quad 4x + 3y + 2z = 1$$

in the field of residues modulo 5 and modulo 7.

1758. Find the largest common divisor of the polynomials

$$f(x) = x^3 + x^2 + 2x + 2, \quad g(x) = x^2 + x + 1$$

(a) over the field of residues modulo 3;
(b) over the field of rational numbers.

1759. Find the largest common divisor of the polynomials

$$f(x) = 5x^3 + x^2 + 5x + 1, \quad g(x) = 5x^2 + 21x + 4$$

(a) over the field of residues modulo 5 (here, each coefficient a is to be understood as the multiple ae of the unit e of the indicated field or the coefficients are to be replaced by their smallest nonnegative residues modulo 5);
(b) over the field of rational numbers.

1760. Find the largest common divisor of the polynomials

$$f(x) = x^4 + 1, \quad g(x) = x^3 + x + 1$$

over the field of residues modulo (a) 3, (b) 5.

*1761. (a) Prove that if the polynomials $f(x)$ and $g(x)$ with integer coefficients are relatively prime over the field Z_p of residues with respect to the prime modulus p, and at least one of the leading coefficients is not divisible by p,

then these polynomials are relatively prime over the field of rational numbers.

(b) Show by way of an example that for any prime p the converse assertion does not hold.

*1762. Prove that the polynomials $f(x)$ and $g(x)$ with integer coefficients are relatively prime over the field of rational numbers if and only if they are relatively prime over the field of residues modulo p, where p is any prime, with the possible exception of a finite set of such numbers.

1763. Factor the polynomial $x^5 + x^3 + x^2 + 1$ into irreducible factors over the field of residues modulo 2.

1764. Factor the polynomial $x^3 + 2x^2 + 4x + 1$ into irreducible factors over the field of residues modulo 5.

1765. Factor the polynomial $x^4 + x^3 + x + 2$ into irreducible factors over the field of residues modulo 3.

1766. Factor the polynomial $x^4 + 3x^3 + 2x^2 + x + 4$ into irreducible factors over the field of residues modulo 5.

1767. Factor all polynomials of degree two in x into irreducible factors over the field of residues modulo 2.

1768. Factor all third-degree polynomials in x into irreducible factors over the field of residues modulo 2.

1769. Find all second-degree polynomials in x with leading coefficient 1 that are irreducible over the field of residues modulo 3.

1770. Find all third-degree polynomials in x with leading coefficient 1 that are irreducible over the field of residues modulo 3.

*1771. Prove that if a polynomial $f(x)$ with integer coefficients is reducible over the field of rational numbers, then it is reducible over the field of residues with respect to any prime modulus p that does not divide the leading coefficient. Give an example of a polynomial that is reducible over the field of rational numbers but is irreducible over the field of residues modulo p, where p divides the leading coefficient.

*1772. Prove that the multiplicative group G of the field Z_p of residues with respect to the prime modulus p is cyclic.

*1773. There exist polynomials with integer coefficients that are irreducible over the field of rational numbers but are reducible over the field of residues with respect to any prime modulus p.

Prove that such, for example, is the polynomial $f(x) = x^4 - 10x^2 + 1$. This polynomial is a polynomial of least degree with integer coefficients that has the root $\alpha = \sqrt{2} + \sqrt{3}$.

*1774. Prove that if all the elements of a commutative ring R have a common divisor a, then the ring has a unit element.

1775. Indicate a commutative ring with unity that contains the element $a \neq 0$ with one of the following properties:

(a) $a^2 = 0$, (b) for a given integer $n > 1$ the following conditions hold: $a^n = 0$, $a^k \neq 0$ if $0 < k < n$.

1776. Let R be a commutative ring with unity e. Prove that (a) the multiplicative inverse (that is, the divisor of unity) cannot be a divisor of zero;

(b) the multiplicative inverse has a unique inverse element;

(c) if δ, ε are inversible, then a is divisible by b if and only if $a\delta$ is divisible by $b\varepsilon$:

(d) the principal ideal (a) of element a in R is different from R if and only if a is not inversible.

1777. Let R be a commutative ring with unity e and without divisors of zero. Prove that

(a) the elements a and b are associated if and only if each of them is divisible by the other;

(b) the principal ideals (a) and (b) coincide if and only if a and b are associated (the definition of a principal ideal is given in problem 1783).

1778. Let R be a commutative ring with unity e and let $R(x)$ be the set of all formal power series $\sum_{n=0}^{\infty} \alpha_n x^n$, $\alpha_n \in R$.

We introduce the ordinary operations of *addition and multiplication of series*:

$$\sum_{n=0}^{\infty} \alpha_n x^n + \sum_{n=0}^{\infty} \beta_n x^n = \sum_{n=0}^{\infty} (\alpha_n + \beta_n) x^n,$$

$$\left(\sum_{n=0}^{\infty} \alpha_n x^n\right)\left(\sum_{n=0}^{\infty} \beta_n x^n\right) = \sum_{n=0}^{\infty} \gamma_n x^n,$$

where $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$.

Show that

(a) $R\langle x\rangle$ is a commutative ring with unity;

(b) $R\langle x\rangle$ contains a subring isomorphic to R;

(c) if R does not have zero divisors, then it also holds for $R\langle x\rangle$;

(d) if R is a field, then $\sum_{n=0}^{\infty}\alpha_n x^n$ is a multiplicative inverse of the ring $R\langle x\rangle$ if and only if $\alpha_0 \neq 0$.

*1779. Let R be the set of all numbers of the form $a + b\sqrt{-3}$, where a and b are rational integers. Show that R is a ring with unity in which factorization into prime factors exists but is not unique. In particular, show that in the two factorizations

$$4 = 2 \cdot 2 = (1+\sqrt{-3})(1-\sqrt{-3})$$

the factors are prime, and 2 is not associated with $1 \pm \sqrt{-3}$.

*1780. Prove that all finite sums $\sum a_i \cdot x^{r_i}$ with real a_i and nonnegative dyadic-rational r_i with respect to the ordinary operations of addition and multiplication of functions form a commutative ring with unity and without zero divisors, in which ring there are no prime elements.

1781. Will the following sets be subgroups of the additive group, subrings, or ideals of the rings indicated below:

(a) the set nZ of numbers that are multiples of $n > 1$, in the ring of integers Z;

(b) the set of integers Z in the ring $Z[x]$ of integral polynomials;

(c) the set $nZ[x]$ of polynomials whose coefficients are multiples of the number $n > 1$, in the ring $Z[x]$ of integral polynomials;

(d) the set N of natural numbers in the ring of integers Z;

(e) the set of integers Z in the ring A of Gaussian integers, that is, numbers of the form $a + bi$ with integral rational a and b;

(f) the set B of numbers $a + bi$, where $a = b$, in the ring A of Gaussian integers;

(g) the set C of numbers of the form $x(1 + i)$ in the ring A of Gaussian integers, where x runs over the entire ring A;

(h) the set $Z[x]$ of integral polynomials in the ring $R[x]$ of polynomials over the field R of rational numbers;

(i) the set I of polynomials not containing terms involving x^k for all $k < n$, where $n > 1$, in the ring $Z[x]$ of integral polynomials;

(j) the set I of polynomials with even constant terms in the ring $Z[x]$ of integral polynomials;

(k) the set I of polynomials with even leading coefficients in the ring $Z[x]$ of integral polynomials.

1782. Prove that the intersection of any set of ideals of a commutative ring R is an ideal.

1783. *The principal ideal* (a) *generated by an element* a *of a commutative ring* R is a minimal ideal containing a. Prove that the ideal (a) exists for any element $a \in R$ and consists of all the elements of the form

(a) ra, where r is any element in R if R has unity;

(b) $ra + na$, where r is any element in R and n is any integer if R does not have a unit element.

1784. A minimal ideal containing M is called the *ideal* (M) *generated by the set* M *of a commutative ring* R. If the set M consists of a finite number of elements $a_1, \ldots, a_s$, then the ideal (M) is also denoted as $(a_1, \ldots, a_s)$. Prove that the ideal (M) exists for any nonempty set $M \in R$ and consists of all finite sums of the form

(a) $\sum r_i a_i$; $r_i \in R$, $a_i \in M$ if R has a unit element;

(b) $\sum r_i a_i + \sum n_i a_i$; $r_i \in R$; $a_i \in M$; n_i are integers if R does not have a unit element.

***1785.** *A ring of principal ideals* is a commutative ring with unity and without zero divisors, in which ring each ideal is a principal ideal (see problem 1783). Prove that each of the following rings is a ring of principal ideals:

(a) the ring of integers, Z;

(b) the ring $P[x]$ of polynomials in the single unknown x over the field P;

(c) the ring A of Gaussian integers.

1786. *The sum of ideals* $I_1, I_2, \ldots, I_k$ *of a commutative ring* R is the set I of all elements x in R that can be represented as

$$x = x_1 + x_2 + \ldots + x_k; \qquad x_i \in I_i; \qquad i = 1, 2, \ldots, k.$$

We write $I = I_1 + I_2 + \ldots + I_k$. If for any x in I the indicated representation is unique, then the sum I is called the *direct sum of ideals* I_i. In this case we write $I = I_1 \dot{+} I_2 \dot{+} \ldots \dot{+} I_k$.

Prove that

(a) the sum of any finite number of ideals is an ideal;

(b) the sum of two ideals is a direct sum if and only if their intersection contains zero alone.

1787. Prove that if $I = I_1 \dot{+} I_2$ is a direct sum of the ideals I_1, I_2, then the product of any element of I_1 by any element of I_2 is equal to zero.

1788. Let $R = I_1 \dot{+} I_2$ be a decomposition of a commutative ring R with unit e into a direct sum of nonzero ideals I_1, I_2.

Prove that if $e = e_1 + e_2$, $e_1 \in I_1$, $e_2 \in I_2$, then e_1, e_2 will be unit elements, respectively, in I_1, I_2 but not in R.

1789. Prove that the factor ring of the ring $D[x]$ of polynomials with real coefficients with respect to the ideal of polynomials divisible by $x^2 + 1$ is isomorphic to the field of complex numbers $a + bi$ with the operations of addition and multiplication defined by the familiar school-textbook rules.

1790. Prove that any homomorphic mapping of a field P into a ring R is either an isomorphic mapping onto some field that enters into R as a subring (a so-called embedding of P in R), or a mapping of all elements of P into the zero of R.

1791. Let Z be a ring of integers and let R be any ring with the unit element e. Prove that the mapping φ, for which $\varphi(n) = ne$, is a homomorphic mapping of Z into R. Find the image $\varphi(Z)$ of the ring Z under this homomorphism.

1792. Let A be the ring of Gaussian integers and let I be the set of all numbers $a + bi$ with even a and b; (a) show that I is an ideal in A, (b) find the cosets of A with respect to I, (c) in the factor ring A/I find the zero divisors and show in that way that A/I is not a field.

1793. Prove that the factor ring A/I of the ring of Gaussian integers with respect to the principal ideal $I = (3)$ is a field of nine elements.

1794. Prove that the factor ring A/I of the ring of Gaussian integers with respect to the principal ideal $I = (n)$ is a field if and only if n is prime and not equal to the sum of two squares of integers.

1795. Let $P[x, y]$ be a ring of polynomials in two unknowns x, y over the field P and let I be the set of all polynomials of this ring without the constant term. Prove that

(a) I is an ideal but is not a principal ideal;

(b) the factor ring $P[x, y]/I$ is isomorphic to the field P.

1796. Let $I = (x, 2)$ be an ideal, generated by a set of two elements x and 2, in the ring of integral polynomials $Z[x]$. Prove that

(a) the ideal I consists of all polynomials with even constant terms;

(b) the ideal I is not a principal ideal;

(c) the factor ring $Z[x]/I$ is isomorphic to the field of residues modulo 2.

1797. Let (n) be an ideal generated by the integer $n > 1$ in the ring of integral polynomials $Z[x]$. Prove that the factor ring $Z[x]/(n)$ is isomorphic to the ring $Z_n[x]$ of polynomials over the ring of residues modulo n.

1798. Let R be the ring of all real functions $f(x)$ defined on the whole number line under the ordinary operations of addition and multiplication and let c be a real number. Prove that

(a) the mapping $\varphi[f(x)] = f(c)$ is a homomorphic mapping of the ring R onto the field D of real numbers;

(b) the kernel of a homomorphism φ, that is, the set I of all elements of the ring R that are mapped into the number 0, is an ideal in R;

(c) the factor ring R/I is isomorphic to the field of real numbers D.

1799. Let Z_p be a field of residues with respect to the prime modulus p and let $f(x)$ be a polynomial of degree n in the ring $Z_p[x]$ that is irreducible over the field Z_p (from field theory we know that such a polynomial exists for any prime p and for any natural n), let I be a principal ideal generated by the polynomial $f(x)$ in the ring $Z_p[x]$. Prove that the factor ring $Z_p[x]/I$ is a finite field and find the number of its elements.

Sec. 22. Modules

A left module over a ring R is an Abelian group M (usually written additively) for the elements of which we define multiplication by elements of R so that $\lambda a \in M$ for arbitrary $\lambda \in R$, $a \in M$, and the following conditions hold (they are similar to the properties of multiplication of a vector by a scalar for a linear space):

1. $\lambda(a + b) = \lambda a + \lambda b$, 2. $(\lambda + \mu) a = \lambda a + \mu a$,

3. $\lambda(\mu a) = (\lambda\mu) a$,

where $\lambda, \mu \in R$, $a, b \in M$.

If, besides, the ring R has a unit element ε and

$$4.\ \varepsilon a = a,\ a \in M,$$

then M is called a *unital left module over* R.

A submodule of a left module M *over a ring* R is a subgroup A of the group M for which $\lambda a \in A$ for arbitrary $\lambda \in R$, $a \in A$.

A mapping φ of the left module M onto the left module M' over one and the same ring R is said to be *homomorphic* if $\varphi(a + b) = \varphi a + \varphi b$, $\varphi(\lambda a) = \lambda \varphi a$ for arbitrary $a, b \in M$, $\lambda \in R$.

A reciprocal one-to-one and homomorphic mapping of a module M onto a module M' (over the same ring) is said to be *isomorphic* (or an *isomorphism*) and the modules M and M' are termed isomorphic.

A factor module M/A *of a left module* M *over a ring* R *with respect to the submodule* A is the factor group M/A with ordinary multiplication by the elements of the ring R: $\lambda(x + A) = \lambda x + A$.

The order $O(a)$ [or the *annihilator* Ann (a)] *of an element* a *of the left module* M *over a ring* R is the set of all elements $\lambda \in R$ for which $\lambda a = 0$. If the order of the element a contains only zero of the ring R, then a is termed the *free element* (or the *element of order zero*). Otherwise a is termed a *periodic element* (or an *element of nonzero order*).

We similarly define right modules M over a ring R with multiplication $a\lambda \in M$, $a \in M$, $\lambda \in R$, and the associated concepts.

When we speak in the following problems of a module M over a ring R, then M is, for the sake of definiteness, to be regarded as a left module over R, although the corresponding properties also hold true for a right module over R.

To simplify matters, some problems are stated for the case of a commutative ring, although they could be generalized to modules over noncommutative rings.

1800. Give examples of a module M over a ring R where we have $\lambda \neq 0$ in R and $a \neq 0$ in M, and $\lambda a = 0$.

1801. Verify that any Abelian group G (written additively) is a module over the ring Z of integers.

1802. The left module M over a ring R is said to be *trivial* if $\lambda a = 0$ for arbitrary $\lambda \in R$, $a \in M$. Prove that the left module M over the ring R with unit element ε can be de-

composed into a direct sum of submodules: $M = M_1 \dot{+} M_2$, where M_1 is unital and M_2 is trivial, M_1 containing all the elements $a \in M$ for which $\varepsilon a = a$ and M_2 containing all elements $a \in M$ for which $\varepsilon a = 0$.

1803. Verify that

(a) if a commutative ring R is regarded as a left module over itself, then the submodules of this module coincide with the ideals of the ring R;

(b) if a noncommutative ring R is regarded as a left (or right) module over itself, then the submodules of that module coincide with the left (or right) ideals of the ring R.

***1804.** Show that an Abelian group G that is primary with respect to the prime number p (problem 1695) may be regarded as a unital module over the ring R of rational numbers whose denominators are not divisible by p.

1805. *A cyclic submodule generated by an element a of the left module M over a ring R* is a minimal submodule $\{a\}$ containing a. Prove that for any $a \in M$ the cyclic submodule $\{a\}$ exists and consists of all elements of the module M having the form:

(a) λa, where $\lambda \in R$ if M is a unital module;

(b) $\lambda a + na$, where $\lambda \in R$ and n is an integer if M is any module.

1806. Prove that an n-dimensional linear space over a field P is (under the same operations) a unital module over P and this module can be decomposed into a direct sum of n cyclic submodules.

1807. Let M be a unital module over a commutative ring R with unit element ε, let $\{a\}$ and $\{b\}$ be cyclic modules, and let $O(a)$ and $O(b)$ be the orders, respectively, of a and b.

(a) Prove that if $\{a\} = \{b\}$, then $O(a) = O(b)$;

(b) use an example to illustrate that the conditions $O(a) = O(b)$ are insufficient for the equality $\{a\} = \{b\}$;

(c) prove that for the equality $\{a\} = \{b\}$ it is necessary and sufficient that $b = \alpha a$, $a = \beta b$, where α, β are certain elements in R;

(d) prove that for the equality $\{a\} = \{b\}$ it is necessary and sufficient that the following conditions hold: $b = \alpha a$, where $a \in R$ and can be inversed with respect to the module $O(a)$; that is, the coset $\alpha + O(a)$ is an inversible element of the factor ring $R/O(a)$.

*1808. Prove that any submodule A of the cyclic module $M = \{a\}$ over the ring of principal ideals of R is itself cyclic.

1809. Let R be the set of all infinite sequences of integers $\alpha = (a_1, a_2, \ldots)$ with addition and multiplication with respect to components. Verify that R is a commutative ring with unity, that is, a cyclic module over itself. In this module, find the submodule that does not have a finite system of generators. This shows that the submodule of a cyclic module need not be cyclic, and the submodule of a finitely generated module need not be finitely generated.

1810. Let M be a module over the commutative ring R.

(a) Prove that if R does not have zero divisors, then the set $A \subset M$ of all periodic elements is a submodule of the module M;

(b) use examples to show that for a ring R with zero divisors the preceding statement may be erroneous.

*1811. Let M be a module over the ring R of principal ideals and let a and b be periodic elements of M of orders $O(a) = (\alpha)$, $O(b) = (\beta)$ and let δ be the greatest common divisor of α and β. Prove that the element $a + b$ is also periodic of order $O(a + b) = (\gamma)$, and (a) γ divides $\frac{\alpha\beta}{\delta}$; (b) γ is a multiple of $\frac{\alpha\beta}{\delta^2}$.

*1812. A module M is said to be *periodic* if all its elements are periodic. The module M over the ring R of principal ideals is said to be *primary* if the orders of all elements in M, being ideals in R, are generated by the powers of one and the same simple element p in R. The module M is said to be a *direct sum of the system* (which is not necessarily finite) of its submodules M_t if each nonzero element in M can be uniquely represented as a sum of a finite number of nonzero elements taken one at a time from certain M_t. Prove that any periodic module M over the ring R of principal ideals can be decomposed into a direct sum of primary submodules.

*1813. Let M be a module over a ring R. Prove the following theorem: in order that any submodule in M can have a finite number of generators, it is necessary and sufficient that in M the condition of maximality for submodules

be satisfied: any increasing sequence of submodules (not necessarily distinct) $M_1 \subseteq M_2 \subseteq \ldots$ becomes stabilized at a finite stage. In particular, this holds for ideals of the ring R if it is regarded as a module over itself.

1814. Prove that if $\{a\}$ and $\{b\}$ are unital cyclic modules over one and the same ring R and the orders of a and b are related by the inclusion $O(a) \subset O(b)$, then there exists a homomorphic mapping of $\{a\}$ onto $\{b\}$.

1815. Let $M = \{a\}$ be a unital cyclic module over a commutative ring R with unit element ε. Prove that

(a) the order $O(a)$ of the element a is an ideal in R:

(b) the factor ring $R/O(a)$, regarded as a module over R with ordinary multiplication defined by multiplication in R, is isomorphic to the module M.

1816. Let $M = A \dotplus B$ be a decomposition of a module M over a ring R into a direct sum of submodules A and B. Prove that the factor module M/A is isomorphic, as a module over R, to the module B.

1817. A module M is said to be an *extension of a module A with the aid of a module B* if A is a submodule of M and the factor module M/A is isomorphic to B (M, A, B are modules over one and the same ring R). Prove that the extension of a finitely generated module with the aid of a finitely generated module is a finitely generated module.

1818. Prove that if the condition of maximality for submodules is fulfilled in a module M (see problem 1813), then this condition is fulfilled in the factor module M/A of the module M with respect to any submodule A.

***1819.** Let A, B be submodules of a module M over a ring R. Prove the following theorem on isomorphism:

$$(A + B)/A \cong B/(A \cap B).$$

***1820.** Let M be a unital module with a finite number of generators $x_1, x_2, \ldots, x_n$ over a commutative ring R with unit element. Prove that if the condition of maximality is fulfilled for ideals in the ring R, then it is fulfilled for submodules in the module M (this theorem can be generalized to noncommutative rings with the maximality condition for left or right ideals).

Sec. 23. Linear Spaces and Linear Transformations (Appendices to Sections 10, 16-19)

1821. Prove that for the equality $\alpha x + \beta y = \beta x + \alpha y$, where α, β are numbers and x, y are vectors, to be fulfilled, it is necessary and sufficient that either $\alpha = \beta$ or $x = y$ hold.

***1822.** (a) Without using the commutativity of addition of vectors, prove that the right inverse and zero elements will be left as well;

(b) using item (a), prove that the commutativity of addition of vectors follows from the other axioms of a linear space.

1823. Let D be the field of real numbers and V the set of all functions specified and assuming positive values on the interval $[a, b]$. We define the addition of two functions and the multiplication of a function by a scalar by the following equalities:

$$f \oplus g = fg, \qquad \alpha \odot f = f^{\alpha}, \qquad f, g \in V, \qquad \alpha \in D.$$

(a) Verify that under the indicated operations, V is a linear space over the field D;

(b) prove that the space V is isomorphic to the space V' of all real functions specified on the interval $[a, b]$ under the ordinary operations of addition of functions and multiplication of a function by a real number;

(c) find the dimension of the space V.

1824. Prove the linear independence of the set of functions $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$, where $\lambda_1, \ldots, \lambda_n$ are pairwise distinct real numbers.

***1825.** Prove the linear independence of the set of functions $x^{\alpha_1}, \ldots, x^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n$ are pairwise distinct real numbers.

***1826.** Prove the linear independence of the following sets of functions:

(a) $\sin x$, $\cos x$;

(b) 1, $\sin x$, $\cos x$;

(c) $\sin x$, $\sin 2x$, $\ldots$, $\sin nx$;

(d) 1, $\cos x$, $\cos 2x$, $\ldots$, $\cos nx$;

(e) 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\ldots$, $\cos nx$, $\sin nx$.

***1827.** Prove the linear independence of the following sets of functions:

(a) $1, \sin x, \sin^2 x, \ldots, \sin^n x$;
(b) $1, \cos x, \cos^2 x, \ldots, \cos^n x$.

1828. Prove the linear dependence of the following sets of functions:

(a) $1, \sin x, \cos x, \sin^2 x, \cos^2 x, \ldots, \sin^n x, \cos^n x$ for $n \geqslant 2$;

(b) $\sin x, \cos x, \sin^2 x, \cos^2 x, \ldots, \sin^n x, \cos^n x$ for $n \geqslant 4$.

***1829.** Verify that all homogeneous polynomials of degree k in n unknowns $x_1, x_2, \ldots, x_n$ with real coefficients (or with coefficients taken from any field) together with zero under ordinary operations form a linear space; find the dimension of the space.

***1830.** Verify that all polynomials of degree $\leqslant k$ in n unknowns $x_1, x_2, \ldots, x_n$ with real coefficients (or coefficients from any field) together with zero form a linear space under ordinary operations; find the dimension of that space.

1831. Let V be a linear space of all polynomials in x of degree $\leqslant n$, $n > 1$, with real coefficients.

(a) Prove that the set L of all polynomials in V having the given real root c is a subspace of V;

(b) find the dimension of L;

(c) the same for the set L_k, $1 \leqslant k \leqslant n$, of all polynomials in V having k distinct real roots $c_1, \ldots, c_k$ (not counting multiplicity);

(d) is the set L' of all polynomials in V having the simple real root c a subspace?

***1832.** Prove the theorem: for two linearly independent systems with the same number of vectors,

$$x_1, \ldots, x_k \tag{1}$$

$$y_1, \ldots, y_k \tag{2}$$

of n-dimensional space V_n to be equivalent (or to generate one and the same subspace), it is necessary and sufficient that in any basis the respective minors of the matrices A and B made up of the coordinate rows of the vectors of the systems be proportional.

1833. Prove that any subspace L of the n-dimensional linear space V_n is a domain of the values of a certain linear transformation φ.

1834. Prove that any subspace L of the n-dimensional linear space V_n is the kernel of some linear transformation φ.

1835. Prove that if for each of the pairwise distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of a linear transformation φ we take a linearly independent system of eigenvectors, then the system containing all the vectors taken is linearly independent.

*1836. Let V be a real linear space (or a linear space over a field P whose characteristic is different from two) and let $V = L \dotplus M$ be a decomposition of the space V into a direct sum of the subspaces L and M. Then any vector x can be uniquely represented in the form

$$x = y + Z; \qquad y \in L, \qquad Z \in M.$$

A linear transformation φ defined by the condition $\varphi x = y$ is called a *projection of the space V on L parallel to M*. According to problem 1538, projections coincide with idempotent transformations, that is with linear transformations having the property $\varphi^2 = \varphi$. Using this fact, prove the following statements:

(a) if φ is a projection on L parallel to M and ε is the identical transformation, then $\varepsilon - \varphi$ is a projection on M parallel to L;

(b) for the sum $\varphi = \varphi_1 + \varphi_2$ of two projections φ_1 and φ_2 to be a projection, it is necessary and sufficient that

$$\varphi_1\varphi_2 = \varphi_2\varphi_1 = \omega, \tag{1}$$

where ω is the zero transformation; if φ_1 and φ_2 are respectively the projections on L_1 parallel to M_1 and on L_2 parallel to M_2 with condition (1) holding, then $\varphi = \varphi_1 + \varphi_2$ is a projection on $L = L_1 \dotplus L_2$ parallel to $M = M_1 \cap M_2$;

(c) for the difference $\varphi = \varphi_1 - \varphi_2$ of two projections φ_1 and φ_2 to be a projection, it is necessary and sufficient that

$$\varphi_1\varphi_2 = \varphi_2\varphi_1 = \varphi_2; \tag{2}$$

if φ_1 and φ_2 are projections defined in item (b) and the condition (2) holds, then $\varphi = \varphi_1 - \varphi_2$ is a projection on $L = L_1 \cap M_2$ parallel to $M = M_1 \dotplus L_2$.

(d) for the product $\varphi = \varphi_1\varphi_2$ of two projections φ_1 and φ_2 to be a projection, it is sufficient that

$$\varphi_1\varphi_2 = \varphi_2\varphi_1; \tag{3}$$

if φ_1 and φ_2 are projections defined in item (b) and the condition (3) holds, then $\varphi = \varphi_1\varphi_2$ is a projection on $L = L_1 \cap L_2$ parallel to $M = M_1 \dotplus M_2$ (here the sum need not necessarily be a direct sum). Use an example to show that the condition (3) is not necessary for the product of the projections φ_1 and φ_2 to be a projection.

*1837. Prove that if for the transformation φ of Euclidean space V into itself there exists a conjugate transformation φ^*, that is a transformation with the property $(\varphi x, y) = (x, \varphi y)$ for any vectors x, y in V, then φ and φ^* are linear. In particular, symmetric and skew-symmetric transformations can be defined, respectively, by the equalities $(\varphi x, y) = (x, \varphi y)$ and $(\varphi x, y) = -(x, \varphi y)$ without the requirement of linearity.

1838. Find the distance between two planes P_1: $x = a_0 + t_1 a_1 + t_2 a_2$ and P_2: $x = b_0 + t_1 b_1 + t_2 b_2$, where $a_0 = (2, 1, 0, 1)$, $a_1 = (1, 1, 1, 1)$, $a_2 = (1, 0, 0, 1)$, $b_0 = (1, -1, -1, 0)$, $b_1 = (1, 1, 0, -1)$, $b_2 = (1, 1, 2, 3)$, and the coordinates of the vectors are given in an orthonormal basis.

1839. A *Hilbert space* is a set V of all infinite sequences of the real numbers $x = (a_1, a_2, \ldots)$ for which the series of squares $\sum_{i=1}^{\infty} a_i^2$ converges. Such sequences are called *vectors* (or *points*) of the space V. Addition of vectors, multiplication of a vector by a scalar, and scalar multiplication of vectors are defined in the usual fashion. Namely, if $x = (a_1, a_2, \ldots)$ and $y = (b_1, b_2, \ldots)$ are vectors and c is a scalar, then

$$x + y = (a_1 + b_1,\ a_2 + b_2,\ \ldots),\quad cx = (ca_1,\ ca_2,\ \ldots),$$

$$(x,\ y) = \sum_{i=1}^{\infty} a_i b_i.$$

Prove that

(a) V is an infinite-dimensional Euclidean space;

(b) if L^* is the orthogonal complement to the subspace L in V (problem 1364), then the equality $V = L \dotplus L^*$ holds true for a finite-dimensional L;

(c) show with examples that the equalities $V = L \dotplus L^*$ and $(L^*)^* = L$ (see problems 1364 and 1365) may not hold for infinite-dimensional subspaces in V.

1840. Let $V_n = L_1 \dot{+} L_2$ be a decomposition of an n-dimensional Euclidean space into a direct sum of two subspaces: L_1^* and L_2^*, which are orthogonal complements of L_1 and L_2 respectively, and let φ be a reflection of V_n in L_1 parallel to L_2. Prove that the transformation φ^*, the conjugate of φ, is a reflection of V_n in L_2^* parallel to L_1^*.

1841. Find all isometric (or orthogonal) transformations that hold the zero vector fixed; (a) in the plane; (b) in three-dimensional space.

***1842.** Find the geometric meaning of a linear transformation φ of three-dimensional Euclidean space specified in the orthonormal basis e_1, e_2, e_3 by the matrix

$$A = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{\sqrt{6}}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & \frac{1}{2} \end{pmatrix}.$$

***1843.** Determine the geometric meaning of a skew-symmetric transformation φ of Euclidean space for the cases: (a) of a straight line; (b) of a plane; (c) of three-dimensional space. Show that in three-dimensional space, φ reduces to the vector premultiplication of all vectors by one and the same vector a, that is, $\varphi x = a \times x$.

***1844.** Prove the following assertion: for a linear transformation φ of Euclidean space (not necessarily finite-dimensional) to be skew-symmetric, it is necessary and sufficient that it carry each vector into the vector orthogonal to it.

Sec. 24. Linear, Bilinear and Quadratic Functions and Forms (Appendix to Sec. 15)

***1845.** Prove that for any nonzero linear function $l(x)$ specified in an n-dimensional linear space V_n there exists a canonical basis in which this function can be written in canonical form $l(x) = x_1$, where x_1 is the first coordinate of the vector x in that basis.

1846. Prove that the nonzero bilinear form $b(x, y) = \sum_{i,j=1}^{n} a_{ij}x_iy_j$ factors into a product of two linear forms

$b(x, y) = l_1(x) l_2(y)$, where $l_1(x) = \sum_{i=1}^{n} b_i x_i$, $l_2(y) = \sum_{j=1}^{n} c_j y_j$, if and only if its rank is equal to unity.

1847. Prove that the bilinear form $b(x, y)$ given in a real n-dimensional space (or in an n-dimensional space over a field with characteristic not equal to two) is symmetric if and only if it has a canonical basis in which it can be written by a bilinear form of canonical aspect: $b(x, y) = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \ldots + \lambda_n x_n y_n$.

***1848.** Prove that if the product of two linear functions specified on a linear space V (not necessarily finite-dimensional) is identically zero, that is, $l_1(x) l_2(x) = 0$ for any $x \in V$, then at least one of the functions is identically zero.

***1849.** Prove that if the symmetric bilinear function $b(x, y)$ specified in a linear space V (not necessarily finite-dimensional) decomposes into two linear functions $b(x, y) = l_1(x) l_2(y)$, then it can be expressed as $b(x, y) = \lambda l(x) l(y)$, where λ is a nonzero number and $l(x)$ is a linear function.

***1850.** Prove that a bilinear function in an n-dimensional real space is of rank 1 if and only if, in some basis, it can be written as a form with the aspect

(a) $\pm x_1 y_1$ if the function is symmetric;

(b) $x_1 y_2$ if the function is nonsymmetric.

***1851.** Prove that a nonzero skew-symmetric function on a linear space V (not necessarily finite-dimensional) cannot decompose into a product of two linear functions.

***1852.** Let $l(x)$ be a nonzero linear function on a linear space V (not necessarily finite-dimensional). Prove that

(a) the kernel S of the function $l(x)$, that is, the set of all vectors $x \in V$ for which $l(x) = 0$, is a maximal linear subspace, that is, S is not contained in a subspace T different from S and V;

(b) for any vector a not in S, any vector x can be uniquely represented as $x = y + \alpha a$, where $y \in S$.

***1853.** Prove that if two linear functions $l_1(x)$ and $l_2(x)$ on a linear space V (not necessarily finite-dimensional) have the same kernel S, then $l_1(x) = \lambda l_2(x)$, where λ is a nonzero number.

***1854.** Using the Jacobi method for computing corner minors, determine the affine class of surfaces in three-

dimensional space:

(a) $x_1^2 + 2x_2^2 - x_3^2 + 2x_1x_2 + 2x_1 + 2x_2 = 0$;

(b) $x_1^2 + x_2^2 + 2x_1x_2 + 2x_1x_3 + 2x_1 + 1 = 0$.

1855. Prove that if a quadratic form with matrix A is positive definite, then so is the quadratic form

(a) with inverse matrix A^{-1};

(b) with the adjoint $\hat{A}$.

***1856.** Let $f(x)$ be a quadratic function on an n-dimensional real linear space V_n. The vector x_0 is said to be a *null vector* if $f(x_0) = 0$. Prove that if the function $f(x)$ is an alternating function, that is, there exist vectors x_1, x_2 such that $f(x_1) > 0$, $f(x_2) < 0$, then there is a basis consisting of null vectors. Suggest a method for constructing such a basis.

***1857.** A *null cone of a quadratic function* $f(x)$ is a set K of all null vectors (problem 1856). Prove that the null cone of a quadratic function $f(x)$ on an n-dimensional real space V_n is a subspace if and only if $f(x)$ is an alternating function, that is, either $f(x) \geqslant 0$ for all x or $f(x) \leqslant 0$ for all x.

***1858.** Let $f(x)$ be a quadratic function on an n-dimensional real linear space V_n, let r be the rank, and p and q the positive and negative indices of inertia of that function. Prove that the maximal dimension of the linear subspaces that enter into the null cone K (problem 1857) is equal to

(a) $\min(p, q)$ if $f(x)$ is nonsingular (that is, $r = n$);

(b) $n - \max(p, q) = \min(p, q) + n - r$ if $f(x)$ is arbitrary (singular or nonsingular).

***1859.** Let $f(x)$ be a quadratic function with the same properties as those in the preceding problem. Prove that the maximal dimension of the linear manifold P that enters into the second-order (quadric) surface S given by the equation $f(x) = 1$ is equal to

(a) $\min(p - 1, q)$ if $f(x)$ is nonsingular;

(b) $\min(p - 1, q) + n - r = n - \max(p, q + 1)$ in the general case.

1860. Use problems 1858 and 1859 to find the maximal dimension of linear manifolds contained in the following quadric surfaces (if the dimension of the space is not indicated, it is assumed to be equal to the largest number-label of the coordinates; the dimension of the empty manifold

is taken to be equal to -1):

(a) $x_1^2 + x_2^2 - x_3^2 = 1$ (hyperboloid of one sheet);

(b) $x_1^2 - x_2^2 - x_3^2 = 1$ (hyperboloid of two sheets);

(c) $x_1^2 - x_2^2 - x_3^2 = 0$;

(d) $x_1 x_2 = 1$;

(e) $x_1 x_2 = 1$ (in three-dimensional space);

(f) $x_1 x_2 = 0$;

(g) $x_1 x_2 = 0$ (in n-dimensional space);

(h) $x_1^2 - x_n^2 = 1$;

(i) $x_1^2 - x_2^2 - \ldots - x_n^2 = 1$;

(j) $x_1^2 + \ldots + x_{n-1}^2 - x_n^2 = 1$;

(k) $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$;

(l) $\sum_{k=1}^{n} (-1)^{k-1} x_k^2 = 1$.

***1861.** The *left kernel* (or *left null space*) of the bilinear function $b(x, y)$ specified on a linear space V is the set L_0' of all vectors $x \in V$ for which $b(x, y) = 0$ for all $y \in V$. The right kernel L_0'' is defined similarly.

Prove that (a) the left and right kernels are subspaces; (b) in an n-dimensional space the left and right kernels have the same dimension $n - r$, where r is the rank of $b(x, y)$, that is the rank of its matrix with respect to some basis.

1862. Find the basis of the left and right kernel (null space) L_0' and L_0'' (problem 1861) for the bilinear form $b(x, y) = x_1 y_1 + 2x_1 y_2 + 3x_2 y_1 + 6x_2 y_2$ and show that $L_0' \neq L_0''$.

***1863.** (a) Prove that for a symmetric and a skew-symmetric bilinear function the left kernel coincides with the right kernel;

(b) give an example of a bilinear function in n-dimensional space which is neither symmetric nor skew-symmetric but for which the left kernel coincides with the right kernel.

***1864.** Prove that a nonzero skew-symmetric bilinear function can be represented in three-dimensional space as $b(x, y) = a(x)b(y) - a(y)b(x)$, where $a(x)$ and $b(x)$ are linear functions.

*1865. Let $b(x, y)$ be a bilinear function and let L be a k-dimensional subspace in an n-dimensional space V_n. Denote by L^* the set of all vectors $y \in V_n$ such that $b(x, y) = 0$ for all $x \in L$. Prove that

(a) L^* is a subspace;

(b) if $b(x, y)$ is nonsingular (that is, the rank is equal to n), then the dimension of L^* is equal to $n - k$;

(c) if $b(x, y)$ has a rank of $r < n$, then the dimension of L^* is greater than or equal to $\max(n - k, n - r)$.

*1866. Let $b(x, y)$ be a nonzero skew-symmetric bilinear function in an n-dimensional linear space V_n. Prove that there is a basis in which $b(x, y)$ can be written by a bilinear form of the following canonical form:

$$b(x, y) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + \ldots + x_{2k-1}y_{2k} - x_{2k}y_{2k-1}, \quad 1 \leqslant k \leqslant \frac{n}{2}.$$

For the following forms find the canonical aspect of the skew-symmetric bilinear form (problem 1866) and the nonsingular transformation of the unknowns that leads to it.

1867. $b(x, y) = x_1y_2 - x_2y_1 + 2(x_1y_3 - x_3y_1) - x_1y_4 + x_4y_1 - 3(x_2y_4 - x_4y_2)$.

1868. $b(x, y) = x_1y_2 - x_2y_1 + 2(x_1y_3 - x_3y_1) + 4(x_2y_4 - x_4y_2)$.

*1869. Let $f(x)$ be a quadratic function in an n-dimensional Euclidean space V_n. Prove the following assertion: for the cone K with equation $f(x) = 0$ (problem 1857) to contain an orthonormal basis of the space V_n, it is necessary and sufficient that the trace of the matrix A of the function $f(x)$ in one (and, hence, in any) orthonormal basis be equal to zero. Formulate this statement in matrix terms.

Sec. 25. Affine (or Point-Vector) Spaces

Definition.[+] Let there be given a set $\mathfrak{A}$ of the elements $A, B, C, \ldots$ (called points) and a linear space V (over the field of real numbers or over any field P) with elements $x, y, z, \ldots$ called vectors. Furthermore, let each ordered pair of points A, B (distinct or coincident) be associated

[+] The definition given here of an affine space is taken (with slight alterations) from N.V. Efimov's *Higher Geometry*, Moscow, 1961 (in Russian).

with a unique vector $x = \overrightarrow{AB}$; two axioms hold relative to this correspondence:

(I) for any point A and any vector x there is a unique point B such that $\overrightarrow{AB} = x$;

(II) for any (not necessarily distinct) three points A, B, C there is an equality that holds true: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

The set $\mathfrak{A}$ together with such a correspondence is termed an *affine space.*

If $V = V_n$ is an n-dimensional linear space, then $\mathfrak{A}$ is also an n-dimensional affine space and is denoted as $\mathfrak{A}_n$. If V is infinite-dimensional, then $\mathfrak{A}$ too is said to be infinite-dimensional. If the linear space V is Euclidean, then the affine space $\mathfrak{A}$ is also termed Euclidean. In this case, the distance between the points A and B is equal to the length of the vector $\overrightarrow{AB}$ and the angle ABC is equal to the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Note. Any linear space V may be regarded as an affine space. Then the set $\mathfrak{A}$ coincides with V so that the vectors are regarded as points too. We also say that the vector x specifies a certain point of affine space. The association of an ordered pair of points of the vector (given in the definition) consists, in this case, in the fact that a vector $z = y - x$ is associated with an ordered pair of points x, y in V whence, via x and z, we can uniquely determine y; this proves axiom I. Axiom II reduces to the obvious equality $(y - x) + (z - y) = z - x$. This is the identification of points and vectors that is assumed in Secs. 16 to 19.

The plane of an affine space $\mathfrak{A}$ *passing through a point* A *and having as its director subspace* L is the set π of all points M in $\mathfrak{A}$ for which the vector $\overrightarrow{AM}$ belongs to L. *The dimension of the plane* π is the dimension of its director subspace L. One-dimensional planes are called *straight lines* and $(n - 1)$-dimensional planes of an n-dimensional space are termed *hyperplanes.*

Two planes π_1 and π_2 are said to be *parallel* (symbolized: $\pi_1 \parallel \pi_2$) if they do not intersect (that is, if they do not have any points in common) and the director subspace of one of them is contained in the director subspace of the other (or the two coincide).

If a certain initial point $O \in \mathfrak{A}$ is chosen, then any point M is uniquely determined by the vector $\overrightarrow{OM}$, and conversely. The vector $\overrightarrow{OM}$ is termed the *radius vector* of the point M. The plane π that passes through the point A and has the director subspace L consists of all points M whose radius vectors are determined from the equation $\overrightarrow{OM} = \overrightarrow{OA} + x$, where $x \in L$. If the vectors in V are regarded as points, then the plane is determined from the equality $\pi = x_0 + L$, where $x_0 = \overrightarrow{OA}$. Thus, in this case the concept of a plane coincides with that of a linear manifold given in Sec. 16. The planes passing through the point O will coincide with the subspaces.

An affine system of coordinates in an n-dimensional affine space $\mathfrak{A}_n$ consists of the point $O \in \mathfrak{A}_n$, called the *origin*, and the basis $e_1, e_2, \ldots, e_n$ of the appropriate linear space V_n. The *coordinates of the point* $M \in \mathfrak{A}_n$ are the coordinates of its radius vector $\overrightarrow{OM}$ in the given basis, that is, the numbers $x_1, x_2, \ldots, x_n$, that satisfy the equality

$$\overrightarrow{OM} = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n.$$

Let a k-dimensional plane π of a real n-dimensional affine space pass through a point A with coordinates $x_1^0, x_2^0, \ldots, x_n^0$ and let it have a director subspace L with a basis consisting of vectors specified by their coordinates:

$$c_i = (c_1^i, c_2^i, \ldots, c_n^i) \quad (i = 1, 2, \ldots, k).$$

Then the coordinates of any point $M \in \pi$ are given by

$$x_i = x_i^0 + t_1 c_i^1 + \ldots + t_k c_i^k \quad (i = 1, 2, \ldots, n). \qquad (1)$$

These equations are called the *parametric equations of the plane* π. The parameters $t_1, t_2, \ldots, t_k$ assume arbitrary real values.

The same plane π may be specified by $n - k$ linearly independent equations of the type

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, n-k). \qquad (2)$$

Here, $b_i = \sum_{j=1}^{n} a_{ij}x_j^0$ and the homogeneous equations $\sum_{j=1}^{n} a_{ij}x_j = 0$ $(i = 1, 2, \ldots, n-k)$ specify the subspace L. We will call the equations (2) the general equations of the plane π.

The equations of a straight line passing through two points $A(x_1^0, x_2^0, \ldots, x_n^0)$ and $B(y_1^0, y_2^0, \ldots, y_n^0)$ are of the form

$$x_i = x_i^0 + t(y_i^0 - x_i^0) \quad (i = 1, 2, \ldots, n). \tag{3}$$

Here, t runs through all the real numbers.

A *segment* AB is a set of points M whose coordinates are obtained from (3) provided that $0 \leqslant t \leqslant 1$. The point M that divides AB in the ratio $\lambda \neq -1$ is determined in vector form by the condition $\overrightarrow{AM} = \lambda \overrightarrow{MB}$ or in coordinates by

$$x_i = \frac{x_i^0 + \lambda y_i^0}{1+\lambda}, \quad i = 1, 2, \ldots, n.$$

***1870.** Given in an affine space four distinct points A, B, C, and D. The points K, L, M, N divide the line segments AB, BC, CD, and DA in the same ratio of $m/n \neq -1$. Prove that

(a) if $ABCD$ is a parallelogram, then $KLMN$ is a parallelogram;

(b) if $KLMN$ is a parallelogram and $m \neq n$, then $ABCD$ is also a parallelogram.

1871. Prove that a pair of coincident points is associated with a zero vector, that is, $\overrightarrow{AA} = O$.

1872. Prove that $\overrightarrow{AB} = -\overrightarrow{BA}$.

1873. Prove that any plane π of affine space is itself an affine space whose dimension is equal to that of π.

1874. Prove that the π-plane passing through a point A with director subspace L does not depend on the choice of A on it, that is, it coincides with the π'-plane passing through point A' from π with the same director subspace L.

Find the parametric equations of the following planes that are specified by general equations:

1875. $\left.\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 1, \\ x_1 + 2x_2 - x_3 + 2x_4 &= 3, \\ x_1 - x_2 - 4x_3 + 5x_4 &= -3. \end{aligned}\right|$

1876. $\begin{aligned} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 &= 2, \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 &= 3, \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 &= 9, \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 &= 1. \end{aligned}$

Find the general equations of the plane specified by the parametric equations:

1877. $\begin{aligned} x_1 &= 2 + t_1 + t_2, \\ x_2 &= 1 + 2t_1 + t_2, \\ x_3 &= -3 + t_1 + 2t_2, \\ x_4 &= 3 + 3t_1 + t_2, \\ x_5 &= 1 + t_1 + 3t_2. \end{aligned}$ **1878.** $\begin{aligned} x_1 &= 1 + t_1 + t_2, \\ x_2 &= 2 + t_2, \\ x_3 &= 5 - t_1 + 3t_2, \\ x_4 &= 3 + 2t_1 - t_2, \\ x_5 &= 1 + 3t_1 - 2t_2. \end{aligned}$

1879. Prove that a unique straight line passes through any two distinct points A and B of affine space.

1880. Prove that there is a unique two-dimensional plane passing through any three points A, B, C of affine space not lying on one straight line.

1881. Prove that a unique k-dimensional plane passes through any $k + 1$ points A_0, A_1, A_2, . . ., A_k of affine space that do not lie in a $(k - 1)$-dimensional plane.

1882. Indicate all cases of mutual positions of three distinct planes in three-dimensional space that are specified by the general equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

and for each case give a necessary and sufficient condition with the aid of the notion of the rank of a matrix.

1883. Using the notion of the rank of a matrix, describe all cases of the mutual positions of two straight lines in three-dimensional space given by the general equations

$$a_1x + b_1y + c_1z = d_1, \qquad a_2x + b_2y + c_2z = d_2,$$

and

$$a_3x + b_3y + c_3z = d_3, \qquad a_4x + b_4y + c_4z = d_4.$$

1884. Using the notion of the rank of a matrix, give all cases of the mutual positions of two two-dimensional planes in four-dimensional space specified by the general equations

$$\sum_{j=1}^{4} a_{ij}x_j = b_i \quad (i = 1, 2) \tag{1}$$

and

$$\sum_{j=1}^{4} a_{ij}x_j = b_i \quad (i = 3, 4). \tag{2}$$

1885. Describe all cases of the mutual positions of two hyperplanes of n-dimensional affine space specified by the general equations

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = c \text{ and}$$

$$b_1x_1 + b_2x_2 + \ldots + b_nx_n = d.$$

***1886.** Prove that if the intersection π of a set (finite or infinite) of planes π_α of affine space is nonempty, then π is a plane.

1887. Prove that two planes $\pi_1 = a_1 + L_1$ and $\pi_2 = a_2 + L_2$ intersect if and only if the vector $a_1 - a_2$ belongs to the subspace $L_1 + L_2$.

***1888.** Prove that a plane π_1, different from a point, of n-dimensional affine space is parallel to any other plane π_2 not intersecting it if and only if π_1 is a hyperplane (that is, if it has the dimension $n - 1$).

***1889.** Prove that two nonintersecting hyperplanes π_1 and π_2 of n-dimensional affine space are parallel if and only if they lie in a plane π_3 having a dimension that is greater by unity than the largest of the dimensions of π_1 and π_2.

***1890.** Prove that parallel hyperplanes π_1' and π_2' can be passed through any nonintersecting planes π_1 and π_2 of an n-dimensional affine space.

1891. Let $\pi_1 = a_1 + L_1$ and $\pi_2 = a_2 + L_2$ be nonintersecting planes of a finite-dimensional affine space. Find the plane π_3 of smallest dimension that contains π_1 and is parallel to π_2.

1892. Prove that if a plane π_0 of affine space is parallel to each of the planes π_α and the intersection π of the planes π_α is nonempty, then π is a plane that is parallel to π_0.

1893. Express the condition of parallelism of two planes in n-dimensional affine space that are specified by general equations via the concept of the rank of a matrix.

1894. A hyperplane specified by the general equation $\sum_{i=1}^{n} a_i x_i = b$ partitions an n-dimensional affine space into two half-spaces consisting of points whose coordinates satisfy one of the inequalities $\sum_{i=1}^{n} a_i x_i \geqslant b$ or $\sum_{i=1}^{n} a_i x_i \leqslant b$. Prove that each of these half-spaces is a convex set.

***1895.**[+] A polyhedron P is given as a convex closure of a system of points of four-dimensional affine space specified by the coordinates:

$$O(0, 0, 0, 0),\ A(1, 0, 0, 0),\ B(0, 1, 0, 0),\ C(1, 1, 0, 0),$$
$$D(0, 0, 1, 0),\quad E(0, 0, 0, 1),\quad F(0, 0, 1, 1);$$

(a) write a system of linear inequalities specifying the polyhedron P;

(b) find all three-dimensional faces of the polyhedron.

1896. Solve the same problem as that of 1895 but for the points

$$O(0, 0, 0, 0),\quad A(1, 0, 0, 0),\quad B(0, 1, 0, 0),$$
$$C(0, 0, 1, 0),\quad D(1, 1, 0, 0),\quad E(1, 0, 1, 0),$$
$$F(0, 1, 1, 0),\quad G(1, 1, 1, 0),\quad H(0, 0, 0, 1).$$

1897. Find in three-dimensional space the vertices and form of a polyhedron P given by the following system of inequalities:

$$x_1 \leqslant 1,\ x_2 \leqslant 1,\ x_3 \leqslant 1,\ x_2 + x_3 \geqslant -1,\ x_1 + x_3 \geqslant -1,$$
$$x_1 + x_2 \geqslant -1.$$

1898. Find the shape and vertices of sections of a four-dimensional cube specified in an orthonormal system of coordinates by the inequalities $0 \leqslant x_i \leqslant 1$, $i = 1, 2, 3, 4$, the sections being cut by the planes

(a) $x_1 + x_2 + x_3 + x_4 = 1$; (b) $x_1 + x_2 + x_3 + x_4 = 2$;

(c) $x_1 + x_2 + x_3 = 1$; (d) $x_1 + x_2 = x_3 + x_4 = 1$.

[+] Problems 1895 to 1898 were suggested by E.B. Vinberg.

*1899. Find, in three-dimensional affine space $\mathfrak{A}_3$ over a field Z_2 consisting of the two elements 0 and 1, (a) the number of all points; (b) the number of all straight lines; (c) the number of all planes; (d) the number of points lying on a single straight line; (e) the number of straight lines passing through one point; (f) the number of points lying in one plane; (g) the number of planes passing through one point; (h) the number of straight lines lying in one plane; (i) the number of planes passing through one straight line; (j) the number of straight lines parallel to a given straight line; (k) the number of planes parallel to a given plane; (l) the number of straight lines parallel to a given plane; (m) the number of planes parallel to a given straight line; (n) the number of straight lines skew to a given straight line.

Sec. 26. Tensor Algebra +

What follows is a brief introduction to the notions and properties that are usually found in any lecture course on the subject. The proof of some of the properties is offered as problems in this section.

Suppose in an n-dimensional linear space V_n (whether real, complex, or over any other field P) there are given two bases $e_1, e_2, \ldots, e_n$ and $e'_1, e'_2, \ldots, e'_n$. These bases are connected by the equalities

$$\begin{aligned} e'_1 &= c_1^1 e_1 + c_1^2 e_2 + \ldots + c_1^n e_n, \\ e'_2 &= c_2^1 e_1 + c_2^2 e_2 + \ldots + c_2^n e_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ e'_n &= c_n^1 e_1 + c_n^2 e_2 + \ldots + c_n^n e_n, \end{aligned}$$

or, more compactly,

$$e'_i = c_i^k e_k \quad (i = 1, 2, \ldots, n). \tag{1}$$

Here and below, we assume summation with respect to the upper or lower index from 1 to n.

+ A number of problems of this section were presented by N.V. Efimov and L.A. Skornyakov in connection with a course of lectures entitled "Linear Algebra and Geometry" that they delivered at the mechanico-mathematics department of Moscow State University from 1964 onwards.

In particular, the definition of a tensor product and the application to it of the concept of a contraction (see introduction and problem 1918) are taken from Efimov's course.

We introduce the transition matrix

$$C = \begin{pmatrix} c_1^1 & c_2^1 & \dots & c_n^1 \\ c_1^2 & c_2^2 & \dots & c_n^2 \\ \dots & \dots & \dots & \dots \\ c_1^n & c_2^n & \dots & c_n^n \end{pmatrix},$$

in the columns of which are the coordinates of the vectors of the second basis with respect to the first basis. If the transition matrix is written in terms of rows (not columns), the formulas that employ matrix multiplication change, while the formulas (1) and (2) and others that do not use matrix multiplication remain unchanged. Then the formulas (1) can be written in matrix form by a single equation:

$$(e_1', e_2', \dots, e_n') = (e_1, e_2, \dots, e_n) C.$$

The coordinates of a vector x in the first basis will be expressed in terms of the coordinates of the same vector in the second basis via the rows of the matrix C using the formulas: $x^k = c_i^k x'^i$, $k = 1, 2, \dots, n$. From this the coordinates x'^i can be expressed in terms of x^k in the form

$$x'^i = d_k^i x^k, \quad i = 1, 2, \dots, n. \tag{2}$$

We introduce the matrix $D = \begin{pmatrix} d_1^1 & d_1^2 & \dots & d_1^n \\ d_2^1 & d_2^2 & \dots & d_2^n \\ \dots & \dots & \dots & \dots \\ d_n^1 & d_n^2 & \dots & d_n^n \end{pmatrix}$, in the columns of which are the coefficients in the expressions x'^i in terms of x^k. Then the formulas (2) can be written down in matrix form by the equality

$$(x'^1, x'^2, \dots, x'^n) = (x^1, x^2, \dots, x^n) D.$$

Since, on the other hand, $(x^1, x^2, \dots, x^n) = (x'^1, x'^2, \dots, x'^n) C^*$, it follows that the matrices C and D are connected by the equality

$$D = (C^*)^{-1}. \tag{3}$$

From now on the asterisk denotes the transpose of a matrix. If both matrices C and D are made up of the coefficients of formulas (1) and (2) not in columns but in rows, then these matrices become the transposes and the equality (3) remains unchanged.

The law of variation similar to that of the basis with respect to formulas (1) is called a *covariant law*, and the law associated with formulas (2) is called a *contravariant law*. Quantities (or other entities) associated with the basis and varying covariantly are termed *covariant*; they are denoted by lower indices; those varying contravariantly are termed *contravariant* and are denoted by upper indices.

A *tensor* in n-dimensional linear space is a correspondence under which to each basis of space there correspond n^{p+q} numbers $a_{i_1 i_2 \ldots i_p}^{j_1 j_2 \ldots j_q}$ labelled with p lower and q upper indices and varying with change of basis covariantly with respect to the lower indices and contravariantly with respect to the upper indices. These numbers are called the *components of the tensor* in the given basis, and the number $p + q$ is termed the *rank* or *order of the tensor*. We also say that a given tensor is p-fold covariant and q-fold contravariant, or is a tensor of type (p, q).

By the definition of a tensor, its components in the two bases

$$e_1, e_2, \ldots, e_n \quad \text{and} \quad e'_1, e'_2, \ldots, e'_n$$

connected by (1) are themselves connected by the equality

$$a'^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} = c^{k_1}_{i_1} c^{k_2}_{i_2} \ldots c^{k_p}_{i_p} d^{j_1}_{l_1} d^{j_2}_{l_2} \ldots d^{j_q}_{l_q} a^{l_1 l_2 \ldots l_q}_{k_1 k_2 \ldots k_p} \qquad (4)$$

$$(i_1, \ldots, i_p, j_1, \ldots, j_q = 1, 2, \ldots, n).$$

As usual, we assume summation from 1 to n over all indices k_s and l_t.

A tensor may be defined differently as a geometric entity associated with a linear space V_n. To do this we consider the conjugate space V_n^*. Its vectors are linear functions $\varphi(x)$ specified on the given space V_n under the ordinary operations of addition of two functions and the multiplication of a function by a scalar. The space V_n^* is also n-dimensional, and to each basis $e_1, e_2, \ldots, e_n$ of the space V_n there corresponds a unique basis $e^1, e^2, \ldots, e^n$ of the conjugate space V_n^*, called the *conjugate* (or *dual*) *basis* for the given basis of the space V_n and connected with it by the equalities

$$e^i(e_j) = \delta^i_j \quad (i, j = 1, 2, \ldots, n), \qquad (5)$$

where δ^i_j is the Kronecker delta equal to 1 when $i = j$ and to 0 when $i \neq j$.

If the basis e_i transforms via formulas (1), then the conjugate basis e^i transforms via the formulas

$$e'^i = d^i_k e^k \quad (i = 1, 2, \ldots, n).$$

The following one-to-one correspondence can be established between the polylinear functions

$$F(x_1, x_2, \ldots, x_p; \varphi^1, \varphi^2, \ldots, \varphi^q)$$

of p vectors in V_n and of q vectors in V_n^* on the one hand and all tensors of the type (p, q) on V_n on the other. This correspondence is isomorphic relative to the operations of addition, multiplication and multiplication by a scalar.

To the above polylinear function there corresponds a tensor whose components in the basis $e_1, e_2, \ldots, e_n$ of the space V_n are defined by the equalities

$$a^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} = F(e_{i_1}, e_{i_2}, \ldots, e_{i_p}; e^{j_1}, e^{j_2}, \ldots, e^{j_q}) \qquad (6)$$

$$(i_1, \ldots, i_p, j_1, \ldots, j_q = 1, 2, \ldots, n).$$

Conversely, associated with a tensor having components $a^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p}$ in the basis $e_1, e_2, \ldots, e_n$ is a polylinear function defined by the equality

$$F(x_1, x_2, \ldots, x_p; \varphi^1, \varphi^2, \ldots, \varphi^q) = a^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} x^{i_1}_1 x^{i_2}_2 \ldots x^{i_p}_p u^1_{j_1} u^2_{j_2} \ldots u^q_{j_q}, \qquad (7)$$

where $x^1_s, x^2_s, \ldots, x^n_s$ are components (or coordinates) of the vector x_s in the basis $e_1, e_2, \ldots, e_n$ and $u^t_1, u^t_2, \ldots, u^t_n$ are components of the vector φ^t in the conjugate basis $e^1, e^2, \ldots, e^n$.

Here, the value of the polylinear function on the given vectors does not depend on the choice of conjugate bases in V_n and V_n^*.

In view of this correspondence, tensors on the space V_n may be defined irrespective of the basis as polylinear functions of the vectors of the space V_n and the conjugate space V_n^*.

A space with a quadratic metric (an *inner-product space*) is an n-dimensional real linear space M_n in which is specified a nonsingular symmetric bilinear function $g(x, y)$ called the metric function. If the quadratic function $g(x, x)$ corresponding to it is positive definite, then the space with the quadratic metric is a Euclidean space and will be denoted

as E_n. In this case, we will write (x, y) instead of $g(x, y)$ and we will call the value of that function the scalar product (inner product) of the vectors x and y.

In the space M_n, to each basis $e_1, e_2, \ldots, e_n$ there corresponds a unique dual basis $e^1, e^2, \ldots, e^n$ associated with the given basis by the equalities

$$g(e_i, e^j) = \delta_i^j \quad (i, j = 1, 2, \ldots, n). \tag{8}$$

Each vector x of the space M_n can be expanded in terms of the bases e_i and e^i: $x = x^i e_i = x_i e^i$. When the basis e_i is changed by means of formulas (1) the components x^i in that basis transform via formulas (2), that is, contravariantly, and are called contravariant components. The dual basis e^i transforms via the formulas

$$e'^i = d_k^i e^k \quad (i = 1, 2, \ldots, n) \tag{9}$$

and the components x_i in this basis transform via the formulas

$$x_i' = c_i^k x_k \quad (i = 1, 2, \ldots, n), \tag{10}$$

that is to say, covariantly, and are called covariant components.

Given a fixed vector $y \in M_n$, the function $\varphi_y(x) = g(x, y)$ is a linear function of x, that is, it is an element of the conjugate space M_n^*. The correspondence $y \to \varphi_y(x)$ is an isomorphic mapping of M_n onto M_n^*. By identifying the elements of these spaces, which correspond to one another under the isomorphism, we may say that the space M_n^*, which is conjugate to the space with the quadratic metric M_n, coincides with M_n. In particular, this holds true for the Euclidean space E_n.

In the inner-product space M_n, one and the same polylinear function of r vectors in M_n may be regarded as a tensor of the type (p, q), where $p + q = r$ and $p = 0, 1, 2, \ldots, r$. For this, choose one of the possible values for p, determine the components of the tensor in the given basis e_i via formulas similar to (6), where e^i is the dual basis connected with the basis e_i by the formulas (8). Conversely, using the given tensor of the type (p, q), we define the value of the corresponding polylinear function on the given vectors by a formula similar to (7). Here, the entities $\varphi^1, \varphi^2, \ldots, \varphi^q$ in the formulas (6) and (7) are to be understood as vectors in the same space M_n.

In particular, to the metric function $g(x, y)$ there correspond two tensors with components

$$g_{ij} = g(e_i, e_j) \quad (i, j = 1, 2, \ldots, n), \tag{11}$$

$$g^{ij} = g(e^i, e^j) \quad (i, j = 1, 2, \ldots, n), \tag{12}$$

which are called, respectively, the covariant metric tensor and the contravariant metric tensor. These tensors are of the type (2, 0) and (0, 2) respectively. Besides, the same function $g(x, y)$ is associated with a third tensor of type (1, 1) whose components are given by the formulas (8) and do not depend on the choice of basis e_i.

The values of the metric function $g(x, y)$ are determined by one of the following three formulas:

$$g(x, y) = g_{ij}x^i y^j, \tag{13}$$

$$g(x, y) = g^{ij}x_i y_j, \tag{14}$$

$$g(x, y) = x_i y^i. \tag{15}$$

In particular, the scalar product (x, y) is computed in similar manner in Euclidean space.

Suppose, in Euclidean space, we consider only orthonormal bases. Then the metric C made up of the coefficients of formulas (1) is orthogonal, from (3) we obtain $D = C$, the matrices made up of the coefficients of formulas (1) and (2) are transposes of each other, the dual basis coincides with the original basis, the covariant components coincide with the contravariant components, and the matrices of the metric tensors g_{ij} and g^{ij} turn into a single matrix. In this case, all tensors of types (p, q) that correspond to the given polylinear function of $p + q$ arguments coincide. The difference between the covariant and contravariant orders of the tensor is of no significance. And for that reason, all indices can be written as subscripts while retaining the required summation signs.

Suppose, in Euclidean space E_n, we have any basis $e_1, e_2, \ldots, e_n$; denote by g the determinant of the matrix $G = (g_{ij})$ of the covariant metric tensor in the given basis. Since $g(x, x)$ is a positive definite quadratic function, it follows that $g > 0$.

The discriminant (covariant) tensor of Euclidean space E_n is a correspondence under which to each basis there corresponds a set of numbers (the components of the tensor)

given by the formulas

$$\varepsilon_{i_1 i_2 \ldots i_n} = (-1)^s \sqrt{g}, \tag{16}$$

where s is the number of inversions in the permutation $i_1, i_2, \ldots, i_n$ and $\varepsilon_{i_1 i_2 \ldots i_n} = 0$ if at least two of the indices are the same. In particular, $\varepsilon_{12\ldots n} = \sqrt{g} > 0$.

When passing to a new basis with the same orientation, the components of a discriminant tensor vary like those of an n-fold covariant tensor; when passing to a basis with opposite orientation, they also change sign.

In similar fashion we define the contravariant discriminant tensor $\varepsilon^{i_1 i_2 \ldots i_n}$.

The oriented volume of a parallelepiped Q constructed on the vectors $x_1, x_2, \ldots, x_n$ of the space E_n is a number (scalar) defined in the given basis e_i by the formula

$$V_e(x_1, x_2, \ldots, x_n) = \sqrt{g} \cdot \det_e(x_1, x_2, \ldots, x_n), \tag{17}$$

where $\sqrt{g} > 0$ and $\det_e(x_1, x_2, \ldots, x_n)$ is the determinant made up of the coordinates (components) of the vectors $x_1, x_2, \ldots, x_n$ in the given basis. Using the discriminant tensor, the oriented volume can be expressed by the formula

$$V_e(x_1, x_2, \ldots, x_n) = \varepsilon_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, \tag{18}$$

where x_i^s is the sth component of the vector x_i in the given basis.

An oriented volume becomes zero if and only if the vectors x_i are linearly dependent. For linearly independent x_i, its absolute value is equal to the volume of the parallelepiped Q, and it is positive or negative depending on whether the set of vectors $x_1, x_2, \ldots, x_n$ is identically or oppositely oriented with basis $e_1, e_2, \ldots, e_n$.

A tensor product. Let V and V' be two linear spaces over one and the same field P. We consider ordered pairs xx' of vectors, where $x \in V$ and $x' \in V'$, and also the formal finite sums of such pairs: $x_1x_1' + \ldots + x_kx_k'$. We now determine the relation of the equivalence of these sums via the following rules: (1) two sums that differ only in the order of the summands are equivalent; (2) the pairs $(\alpha x)\, x'$ and $x\, (\alpha x')$, where $\alpha \in P$, are equivalent; (3) the pair $(x + y)\, x'$ is equivalent to the sum of the pairs $xx' + yx'$ and the pair $x\, (x' + y')$ is equivalent to the sum of the pairs $xx' + xy'$.

Two sums are equivalent if we can pass from one of them to the other by applying the indicated rules to the whole sum or to a part of it a finite number of times. This relation of equivalence is reflexive, symmetric and transitive. For this reason, all sums can be separated into classes of equivalent sums. Let T be the set of these classes. We introduce into T the operations of addition and multiplication by elements of the field P in terms of operations over the representatives of the classes. The *sum* of two sums is a sum obtained by adjoining to the first sum the terms of the second sum. *Multiplication* of a sum by $\alpha \in P$ is defined as the multiplication by α of the first elements of all pairs of the given sum. For example, $\alpha(xx' + yy') = (\alpha x)x' + (\alpha y)y'$.

Under these operations, the set T is a linear space over the field P. It is termed a *tensor product of the spaces* V and V' and is denoted as $V \times V'$. If V and V' are finite-dimensional, then $V \times V'$ is also finite-dimensional and its dimension is equal to the product of the dimensions of V and V'. If $e_1, e_2, \ldots, e_n$ is a basis of V and $e'_1, e'_2, \ldots, e'_{n'}$ is a basis of V', then the system of classes of equivalent sums containing the ordered pairs

$$e_i e'_j \ (i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n') \qquad (19)$$

is a basis of the space $V \times V'$ (see problem 1918).

In similar fashion we can define a tensor product of any finite set of linear spaces.

Tensors of type (p, q) specified on a space V_n may be regarded as vectors of a space T, which is a tensor product q of spaces V_n and p of the conjugate spaces V_n^*. To do this, we take the basis $e_1, e_2, \ldots, e_n$ in V_n and the conjugate basis $e^1, e^2, \ldots, e^n$ in V_n^*. Then the ordered sets

$$e_{j_1} e_{j_2} \ldots e_{j_q} e^{i_1} e^{i_2} \ldots e^{i_p}, \qquad (20)$$

where all indices vary from 1 to p, constitute a basis of the space T. A vector t in T is expressed in terms of this basis as

$$t = a^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} e_{j_1} e_{j_2} \ldots e_{j_q} e^{i_1} e^{i_2} \ldots e^{i_p}.$$

Its components, that is, the numbers $a^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p}$, will be components of a (p, q)-type tensor in the basis $e_1, e_2, \ldots, e_n$ in the meaning of the first definition of a tensor.

1900. Prove that for any basis $e_1, \ldots, e_n$ of an n-dimensional linear space V_n there is a unique conjugate basis $e^1, \ldots, e^n$ of the conjugate space V_n^*, that is, a basis connected with the given basis by the conditions $e^j(e_i) = \delta_i^j$ $(i, j = 1, 2, \ldots, n)$, where δ_i^j is the Kronecker delta.

1901. A linear function $\varphi(x)$ on an n-dimensional linear space V_n is a covariant tensor, that is, a tensor of the type (1, 0).

(a) Find its components a_i in the given basis e_i.

(b) Show that the numbers a_i are components of $\varphi(x)$ as of a vector of the conjugate space V_n^* in the basis e^i which is conjugate to the basis e_i of V_n.

1902. The components (coordinates) of the vector x in a given basis of the n-dimensional linear space V_n determine a contravariant tensor, that is, a tensor of the type (0, 1). Write this tensor in the form of a polylinear function.

1903. Let $\varphi(x) = a_1x^1 + a_2x^2 + \ldots + a_nx^n$ be a linear form of the components of the vector $x \in V_n$ in the basis e_i. Show that the coefficients a_ia_j $(i, j = 1, 2, \ldots, n)$ of the square of the form yield a twice covariant tensor.

1904. We say the following matrix is the matrix of a twice covariant tensor a_{ij} in a given basis:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Find the law of variation of the matrix A when passing to a new basis.

1905. We say the following matrix is the matrix of a twice contravariant tensor a^{ij} in a given basis:

$$A = \begin{pmatrix} a^{11} & a^{12} & \ldots & a^{1n} \\ a^{21} & a^{22} & \ldots & a^{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a^{n1} & a^{n2} & \ldots & a^{nn} \end{pmatrix}.$$

Find the law of variation of the matrix A when passing to a new basis.

1906. Let a_i^j be a tensor of the type (1, 1) in an n-dimensional linear space V_n. Assuming i to be the number of

a column and j to be the number of a row, we obtain a matrix made up of the components of the tensor:

$$A = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^n & a_2^n & \dots & a_n^n \end{pmatrix}.$$

Find the law of variation of the matrix A when passing to a new basis.

1907. (a) Show that the Kronecker delta δ_i^j yields a tensor of type (1, 1) in all bases of an n-dimensional linear space; (b) write this tensor in the form of a polylinear function.

***1908.** Let a_{ij} be a twice covariant tensor in an n-dimensional space. Prove that

(a) if the matrix A of the tensor a_{ij} is nonsingular in one basis, then it is nonsingular in any basis;

(b) the elements a^{ij} of the inverse of A of the tensor a_{ij} (for the case when A is nonsingular) form a twice contravariant tensor.

1909. Let Ax be a linear transformation and let $\varphi(x)$ be a linear function on an n-dimensional linear space V_n. Show that the function $F(x; \varphi) = \varphi(Ax)$ is a tensor of type (1, 1), the matrix of which in any basis coincides with the matrix of the linear transformation Ax in the same basis.

1910. Show that the elements of the matrix of the bilinear function $F(x, y)$ in an n-dimensional linear space form a tensor of type (2, 0), that is to say, a twice covariant tensor.

1911. Prove that the elements of the matrix of a linear transformation in a given basis form a tensor of type (1, 1).

1912. Given in an n-dimensional linear space V_n p vectors $x_1, x_2, \dots, x_p$. Write down the coordinates (components) of the vectors in some basis according to the rows of the matrix

$$\begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^n \\ a_2^1 & a_2^2 & \dots & a_2^n \\ \dots & \dots & \dots & \dots \\ a_p^1 & a_p^2 & \dots & a_p^n \end{pmatrix}.$$

(a) Show that the numbers

$$a^{i_1 i_2 \ldots i_p} = \begin{vmatrix} a_1^{i_1} & a_1^{i_2} & \ldots & a_1^{i_p} \\ a_2^{i_1} & a_2^{i_2} & \ldots & a_2^{i_p} \\ \ldots & \ldots & \ldots & \ldots \\ a_p^{i_1} & a_p^{i_2} & \ldots & a_p^{i_p} \end{vmatrix} \quad (i_1, i_2, \ldots, i_p = 1, 2, \ldots, n)$$

are components of a p-fold contravariant tensor, that is, a tensor of type $(0, p)$ in the given basis. This tensor is termed a *p-vector* (for $p = 1$ it is a *vector*, for $p = 2$ it is a *bivector*).

(b) Prove that a *p-vector* is zero if and only if the given vectors $x_1, x_2, \ldots, x_p$ are linearly dependent (in particular, when $p > n$, all p-vectors are zero).

(c) Prove that two linearly independent sets, each made up of p-vectors, are equivalent if and only if the corresponding p-vectors differ in a nonzero factor.

(d) Show that if tensors are regarded as polylinear functions (see the introduction to this section), then a p-vector specified by the vectors $x_1, x_2, \ldots, x_p$ of the space V_n can be defined as a polylinear function of p-vectors $\varphi^1, \varphi^2, \ldots, \varphi^p$ of the conjugate space V_n^* specified by the equation

$$F(\varphi^1, \varphi^2, \ldots, \varphi^p) = \begin{vmatrix} \varphi^1(x_1) & \varphi^2(x_1) & \ldots & \varphi^p(x_1) \\ \varphi^1(x_2) & \varphi^2(x_2) & \ldots & \varphi^p(x_2) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi^1(x_p) & \varphi^2(x_p) & \ldots & \varphi^p(x_p) \end{vmatrix}.$$

1913. Determine how the components of a tensor of type (p, q) vary when passing from a basis $e_1, \ldots, e_n$ to the basis $e_1', \ldots, e_n'$ obtained from the preceding one by a given substitution $\pi(i) = k_i$ $(i = 1, 2, \ldots, n)$. This means that $e_i' = e_{\pi(i)}$ $(i = 1, 2, \ldots, n)$.

1914. Relative to the basis $e_1, \ldots, e_n$, find the components of a tensor of type (n, n) given by the equation

$$F(x_1, \ldots, x_n; \varphi^1, \ldots, \varphi^n) = \begin{vmatrix} \varphi^1(x_1) & \varphi^2(x_1) & \ldots & \varphi^n(x_1) \\ \varphi^1(x_2) & \varphi^2(x_2) & \ldots & \varphi^n(x_2) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi^1(x_n) & \varphi^2(x_n) & \ldots & \varphi^n(x_n) \end{vmatrix}.$$

1915. Let $F(x_1, \ldots, x_n; \varphi^1, \ldots, \varphi^n)$ be a polylinear function that is skew-symmetric both in the arguments $x_1, \ldots, x_n$ and in the arguments $\varphi^1, \ldots, \varphi^n$. Prove that its values can be expressed in terms of the components in the given basis by the formula

$$F(x_1, \ldots, x_n; \varphi^1, \ldots, \varphi^n) = \det(x) \det(\varphi) \cdot F(e_1, \ldots, e_n; e^1, \ldots, e^n),$$

where e^i is the conjugate of the basis e_i, $\det(x)$ is the determinant made up of the components of the vectors $x_1, \ldots, x_n$ in the basis $e_1, \ldots, e_n$, and $\det(\varphi)$ is the determinant made up of the components of the vectors $\varphi^1, \ldots, \varphi^n$ in the basis $e^1, \ldots, e^n$.

1916. Given a tensor of the type (p, q) in the form of a polylinear function $F(x_1, \ldots, x_p; \varphi^1, \ldots, \varphi^q)$. Its *contraction* with respect to the numbers $k \leqslant p$, $l \leqslant q$ is the sum

$$\sum_{i=1}^{n} F(x_1, \ldots, x_{k-1}, e_i, x_{k+1}, \ldots, x_p; \varphi^1, \ldots, \varphi^{l-1}, e^i, \varphi^{l+1}, \ldots, \varphi^q).$$

Show that the contraction does not depend on the basis and is a tensor of the type $(p-1, q-1)$. It is assumed that $p, q > 0$.

1917. Given a symmetric tensor a_{ij} and a skew-symmetric tensor b^{ij}. Find their complete contraction $a_{ij}b^{ij}$.

***1918.** In studying a tensor product (see the introduction to this section), it is convenient to use the concept of a contraction in the following meaning. For the sake of simplicity, let us consider the tensor product of two linear spaces: $T = V \times V'$, where V and V' are linear spaces over the same field P.

The contraction of the sum $x_1x_1' + \ldots + x_kx_k'$, where $x_i \in V$, $x_i' \in V'$ $(i = 1, 2, \ldots, k)$, with the vector φ of the space V^*, which is the conjugate of V, is the vector $\alpha_1x_1' + \ldots + \alpha_kx_k' \in V'$, where $\alpha_i = \varphi(x_i) \in P$ $(i = 1, 2, \ldots, k)$. Similarly we can define the contraction of the same sum with the vector $\varphi' \in V'^*$; it is a vector in V.

(a) Prove that if two sums of the above type are equivalent, then their contractions are equal. This permits determining the contraction $\varphi(t) \in V'$ for $t \in T$, $\varphi \in V^*$.

(b) For a pair xx' to be equivalent to the pair $00'$, where 0 and $0'$ are zero vectors taken from V and V' respectively, it is necessary and sufficient that at least one of the following conditions hold: $x = 0$, $x' = 0'$.

(c) If $x_1x_1' + \ldots + x_kx_k' \sim 00'$ and the vectors $x_1', \ldots, x_k'$ are linearly independent, then

$$x_1 = \ldots = x_k = 0.$$

(d) The classes of equivalent sums containing the pairs e_ie_j' $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n')$, where $e_1, \ldots, e_n$ is a basis in V and $e_1', \ldots, e_{n'}'$ is a basis in V', form a basis T.

1919. Prove that the tensor products of linear spaces of all polynomials in x and all polynomials in y with coefficients in the field P is a space of all polynomials in the two unknowns x, y with coefficients in P.

1920. Prove that the tensor product of the space of polynomials $f(x)$ of degree $\leqslant n$ and of the space of the polynomials $g(y)$ of degree $\leqslant s$ with coefficients in the field P is a space of polynomials $h(x, y)$ of degree $\leqslant n$ in x and $\leqslant s$ in y with coefficients taken from the field P.

1921. Prove the following statements:

(a) for any basis e_i of an n-dimensional Euclidean space E_n (or an inner-product space M_n) there is a unique dual basis e^i of the space E_n (respectively, of M_n) that is connected with the given basis by the equalities $(e_i, e^j) = \delta_i^j$ $(i, j = 1, 2, \ldots, n)$ [respectively, by the equalities (8) of the introduction to this section];

(b) the dual basis can be expressed in terms of the given basis via the formulas $e^i = g^{i\alpha}e_\alpha$ $(i = 1, 2, \ldots, n)$;

(c) the given basis can be expressed in terms of the dual basis via the formulas $e_i = g_{i\alpha}e^\alpha$ $(i = 1, 2, \ldots, n)$;

(d) the metric tensors g_{ij} and g^{ij} are connected by the equalities $g_{i\alpha}g^{j\alpha} = \delta_i^j$ $(i, j = 1, 2, \ldots, n)$;

(e) if we denote the matrices of the metric tensors by $G = (g_{ij})_1^n$ and $G_1 = (g^{ij})_1^n$, and the determinants of these matrices by $g = |g_{ij}|_1^n$ and $g_1 = |g^{ij}|_1^n$, then $GG_1 = E$, $gg_1 = 1$.

1922. Prove that the covariant and contravariant components of one and the same vector in a given basis of Euclidean space are connected by the following equalities:

(a) $x_i = g_{i\alpha}x^{\alpha}$ $(i = 1, 2, \ldots, n)$;

(b) $x^i = g^{i\alpha}x_{\alpha}$ $(i = 1, 2, \ldots, n)$.

1923. Two vectors are given in a rectangular Cartesian coordinate system:

$$e_1 = (1, 0), \quad e_2 = (\cos\alpha, \sin\alpha), \quad \sin\alpha \neq 0;$$

(a) verify that these vectors form a basis;
(b) find the covariant metric tensor g_{ij} in the basis e_1, e_2;
(c) find the contravariant metric tensor g^{ij} in the basis e_1, e_2;
(d) find an expression for the vectors of the dual basis e^1, e^2 in terms of the basis e_1, e_2 and their components in the original rectangular system;
(e) write down an expression of the scalar product (x, y) of two vectors in terms of their components in the basis e_1, e_2;
(f) find the discriminant tensor ε_{ij} in the basis e_1, e_2;
(g) write down an expression for the oriented area of a parallelogram, constructed on the vectors x, y, in terms of the discriminant tensor ε_{ij} and in terms of the determinant made up of the components in the basis e_1, e_2.

***1924.** Let e_1, e_2 be any basis of a plane, let g^{ij} be the contravariant metric tensor, and let ε_{ij} be the discriminant tensor in that basis. Using the given vector $x = x^i e_i$, one constructs a vector $y = y^i e_i$, where $y^i = g^{ij}\varepsilon_{jk}x^k$. Determine to what extent the vector y depends on the choice of basis. What is the geometric relationship between the vectors x and y?

1925. In a four-dimensional Euclidean space we have a twice contravariant tensor a^{kl}. How do the quantities $b_{ij} = \varepsilon_{ijkl}a^{kl}$ $(i, j = 1, 2, 3, 4)$ change with a change of basis?

***1926.** Suppose we have, in some basis of an n-dimensional Euclidean space E_n, a tensor a_{ijk} of the type (3, 0) and suppose g_{ij} and g^{ij} are metric tensors. Determine the tensor a^{ij}_k of the type (1, 2) by the equalities $a^{ij}_k = g^{i\alpha}g^{j\beta}a_{\alpha\beta k}$ $(i, j, k = 1, 2, \ldots, n)$. Express a_{ijk} in terms of a^{ij}_k.

1927. A three-dimensional Euclidean space is defined by a metric tensor with the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 3 & 10 & 1 \\ 1 & 1 & 6 \end{pmatrix}$. Find the lengths of line segments cut off on the coordinate axes by the plane $\frac{x^1}{2}+\frac{x^2}{3}+\frac{x^3}{5}=1$.

***1928.** The metric of a three-dimensional Euclidean space is given by the metric tensor g_{ij} with matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 6 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Find the area S of a triangle with vertices $A(1, 0, 1)$, $B(2, 1, 1)$, $C(3, 1, 2)$ and height h dropped from C onto AB if the coordinates and the tensor are given in the same basis.

***1929.** A three-dimensional Euclidean space is given by the metric tensor g_{ij} with the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{pmatrix}$. Find the foot of the perpendicular PQ dropped from the point $P(1, -1, 2)$ on the plane $x^1+x^2+2x^3+2=0$.

***1930.** In a four-dimensional Euclidean space E_4 we are given three vectors x, y, z and we have constructed a vector u with covariant components $u_i = \varepsilon_{ijkl}x^j y^k z^l$ $(i = 1, 2, 3, 4)$, where ε_{ijkl} is the discriminant tensor in the same basis (for a four-dimensional analogue of a vector product see problem 1938). Prove that

(a) the vector u is orthogonal to each of the vectors x, y, z;

(b) if the vectors x, y, z are linearly independent, then the length $|u|$ of the vector u is equal to the three-dimensional volume $V(x, y, z)$ of a parallelepiped constructed on x, y, z; but if x, y, z are linearly dependent, then $u = 0$.

1931. Find the complete contraction $g_{ij}g^{ij}$ of metric tensors in the n-dimensional Euclidean space E_n.

1932. We determine the contravariant discriminant tensor of the n-dimensional Euclidean space E_n by the formulas $\varepsilon^{i_1 i_2 \ldots i_n} = (-1)^s \sqrt{g_1}$, where $g_1 = |g^{ij}|_1^n$, s is the number of inversions in the permutation $i_1, i_2, \ldots, i_n$, and $\varepsilon^{i_1 i_2 \ldots i_n} = 0$ if at least two of the indices $i_1, i_2, \ldots, i_n$ coincide. Prove that

(a) an oriented volume of the parallelepiped constructed on the vectors $x_1, x_2, \ldots, x_n$ (see the introduction to this section) can be expressed by the equalities

$$V_e(x_1, x_2, \ldots, x_n) = \sqrt{g_1}\det_1(x_1, x_2, \ldots, x_n) = \varepsilon^{i_1, i_2, \ldots, i_n} x^1_{i_1} x^2_{i_2} \ldots x^n_{i_n},$$

where $\det_1(x_1, x_2, \ldots, x_n)$ is the determinant made up of the covariant components of the vectors $x_1, x_2, \ldots, x_n$ and x^s_i is the ith covariant component of the vector x_s;

(b) $$\varepsilon^{i_1 i_2 \ldots i_n} = g^{i_1\alpha_1} g^{i_2\alpha_2} \ldots g^{i_n\alpha_n} \varepsilon_{\alpha_1\alpha_2\ldots\alpha_n};$$

(c) $$\varepsilon_{i_1 i_2 \ldots i_n} = g_{i_1\alpha_1} g_{i_2\alpha_2} \ldots g_{i_n\alpha_n} \varepsilon^{\alpha_1\alpha_2\ldots\alpha_n}.$$

1933. Compute the contraction of the discriminant tensors $\varepsilon_{ijk}\varepsilon^{ijk}$ of three-dimensional Euclidean space.

1934. In the basis e_1, e_2, e_3 of three-dimensional real space we are given the covariant metric tensor g^{ij} with the matrix

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

(a) Verify that the space is Euclidean;

(b) find the matrix G_1 of the contravariant metric tensor g^{ij};

(c) find the contravariant components of the unit vector of the normal to the plane given in the same basis by the equation $3x^1 + 2x^2 - 3x^3 - 5 = 0$.

***1935.** Prove that the square of an oriented volume of a parallelepiped constructed on n vectors of n-dimensional Euclidean space is equal to the Gramian of the vectors.

1936. Given, in an n-dimensional Euclidean space, the hyperplane π specified by the equation $a_1x^1 + \ldots + a_nx^n + b = 0$. Also given is a point $M(x^1_0, \ldots, x^n_0)$.

(a) Show that the vector p with covariant components $a_1, \ldots, a_n$ is perpendicular to the plane π;

(b) Show that the distance d from the point M to the plane π can be expressed by the formula $d = \frac{|a_1x^1_0 + \ldots + a_nx^n_0 + b|}{|p|}$.

***1937.** Find the distance from the point $M(x_0, y_0)$ on a Euclidean plane to the straight line $ax + by + c = 0$

if the components are given in the basis

$$e_1 = \{1, 0\}, \quad e_2 = \{\cos \omega, \sin \omega\}, \quad \sin \omega \neq 0,$$

(the components of the vectors of the basis are given in a rectangular Cartesian coordinate system).

***1938.** Let ε_{ijk} be the discriminant tensor of a three-dimensional Euclidean space, and let x^i, y^i be the contravariant components of the vectors x, y in some basis. Prove that the vector z, the covariant components of which in the same basis are given by the formulas $z_i = \varepsilon_{i\alpha\beta}x^\alpha y^\beta$ $(i = 1, 2, 3)$, is a vector product of x and y.

Answers

Chapter 1. Determinants

1. 1. 2. —2. 3. —1. 4. 0. 5. 0. 6. —1. 7. $4ab$. 8. $-2b^3$. 9. 1. 10. $\sin(\alpha - \beta)$. 11. $\cos(\alpha + \beta)$. 12. 0. 13. 1. 14. 1. 15. —1. 16. 1. 17. 0. 18. $ab - c^2 - d^2$. 19. $(a - b)^2$. 20. 0. 21. $a^2 + b^2 + c^2 + d^2$. 22. $x = 3$; $y = -1$. 23. $x = 5$; $y = 2$. 24. $x = \frac{2}{3}$; $y = \frac{1}{3}$. 25. $x = 2$; $y = -3$. 26. $x = \cos(\beta - \alpha)$; $y = \sin(\beta - \alpha)$. 27. $x = \cos\alpha\cos\beta$; $y = \cos\alpha\sin\beta$.

28. The system is indeterminate; Cramer s formulas do not yield a correct answer since, by these formulas, x and y are equal to $\frac{0}{0}$, that is to say, they can assume arbitrary values, whereas they are connected by the relation $2x + 3y = 1$, whence the value of one unknown determines a unique value of the other.

29. The system is inconsistent.

30. For $a \neq b$ the equation is evendetermined, for $a = b \neq c$ it is inconsistent, and for $a = b = c$ it is indeterminate.

31. For $\alpha = k\pi$, where k is an integer, the equation is inconsistent, and for all other values of α it is evendetermined.

32. For $\alpha = 2k\pi$, where k is an integer, the equation is inconsistent, for $\alpha = (2k + 1)\pi$ it is indeterminate, and for all other values of α it is evendetermined.

33. For $\alpha + \beta \neq k\pi$, where k is an integer, the equation is evendetermined, for $\alpha + \beta = 2k\pi$ and for $\alpha + \beta = (2k_1 + 1)\pi$, $\alpha = k_2\pi$, where k_1 and k_2 are integers, it is indeterminate, and for $\alpha + \beta = (2k_1 + 1)\pi$, $\alpha \neq k_2\pi$ it is inconsistent.

34. For $a \neq 0$ the system is evendetermined, for $a = b = 0$ it is indeterminate, and for $a = 0 \neq b$ it is inconsistent.

35. For $ac - b^2 \neq 0$ the system is evendetermined, for $ac - b^2 = 0$ it is indeterminate. It cannot be inconsistent.

36. For $a \neq \pm 6$ the system is evendetermined, for $a = 6$ it is indeterminate, and for $a = -6$ it is inconsistent.

37. For $ab \neq 90$ the system is evendetermined, for $a = 6$, $b = 15$ it is indeterminate, for $ab = 90$ but $a \neq 6$, $b \neq 15$ it is inconsistent.

39. *Hint.* Be sure that in the quadratic equation formula the radicand is a positive expression.

40. *Solution.* Suppose a given trinomial is a perfect square, that is, $ax^2 + 2bx + c = (px + q)^2$. Comparing the coefficients of identical powers of x, we find $a = p^2$, $b = pq$, $c = q^2$, whence $ac - b^2 = p^2q^2 - (pq)^2 = 0$. Suppose, conversely, $ac - b^2 = 0$.

Then $ax^2+2bx+c=\frac{1}{a}(a^2x^2+2abx+ac)=\frac{1}{a}[(ax+b)^2+(ac-b^2)]=$ $=\frac{1}{a}(ax+b)^2$ is a perfect square, since a square root may be taken of the complex number $\frac{1}{a}$.

42. *Solution.* If $\frac{ax+b}{cx+d}=q$ for any x, then $ax+b=q(cx+d)$, $a=qc$, $b=qd$ and $ad-bc=0$. Conversely, if $ad-bc=0$, then for $c\neq 0\neq d$ we have $\frac{a}{c}=\frac{b}{d}=q$, $a=qc$, $b=qd$. For $c=0\neq d$ we have $a=0$ and, assuming $q=\frac{b}{d}$, we again have $a=qc$, $b=qd$. For $c\neq 0=d$ we get the same, assuming $q=\frac{a}{c}$. Therefore $\frac{ax+b}{cx+d}=$ $=\frac{q(cx+d)}{cx+d}=q$ for any x.

43. 40. 44. -3. 45. 100. 46. -5. 47. 0. 48. 1. 49. 1. 50. 2. 51. 4. 52. -8. 53. 6. 54. 20. 55. 0. 56. $3abc-a^3-b^3-c^3$. 57. a^3+b^3+ $+c^3-3abc$. 58. 0. 59. $2x^3-(a+b+c)x^2+abc$. 60. $(ab+$ $+bc+ca)x+abc$. 61. $1+\alpha^2+\beta^2+\gamma^2$. 62. 1. 63. $\sin(\beta-\gamma)+$ $+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$. 64. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. 66. -2.

67. $xyz+2(ace-bcf+adf+bde)-x(e^2+f^2)-y(c^2+d^2)-$ $-z(a^2+b^2)$. 68. 0. 69. -3. 70. $3i\sqrt{3}$.

72. 4. *Hint.* All six terms of the determinant cannot be equal to $+1$, for then the product of three terms $a_{11}a_{22}a_{33}$, $a_{12}a_{23}a_{31}$, $a_{13}a_{21}a_{32}$ would be equal to the product of the three remaining terms, whereas the first of the products is equal to the product of all nine elements of the determinant, and the second is equal to the same product of the nine elements with sign reversed. Furthermore, satisfy yourself that the determinant is different from 5 and that

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}=4.$$

73. 2. *Hint.* Show that all three positive terms that appear in the determinant cannot be equal to 1 and take into account that

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}=2.$$

74. $x=3$, $y=-2$, $z=2$. 75. $x=y=z=1$. 76. $x=1$, $y=2$, $z=-1$. 77. $x=2$, $y=-3$, $z=-2$. 78. $x=-a$, $y=$ $=b$, $z=c$. *Hint.* Set. $\frac{x}{a}=x'$; $\frac{y}{b}=y'$; $\frac{z}{c}=z'$.

79. $x=bc$, $y=ac$, $z=ab$. 80. $x=a$, $y=2b$, $z=3c$. *Hint.* Divide each of the equations by abc and set $\frac{x}{a}=x'$; $\frac{y}{b}=y'$; $\frac{z}{c}=z'$.

81. $x = \frac{a+b+c}{3}$, $y = \frac{a+b\varepsilon^2+c\varepsilon}{3}$, $z = \frac{a+b\varepsilon+c\varepsilon^2}{3}$. *Hint.* This system can be solved via the Cramer formulas. It is simpler first to add all the equations and then add after multiplying the second equation by ε^2, and the third by ε, and, finally, add after multiplying the second equation by ε and the third by ε^2. Use the relation $1 + \varepsilon + \varepsilon^2 = 0$.

82. The system is indeterminate since the third equation is the sum of the other two and hence any solution of the first two equations satisfies the third one as well. The first two equations have an infinity of solutions, for example, x and y can be expressed in terms of z thus: $x = 10z + 1$, $y = 7z$. We find the values of x and y by assigning an arbitrary value to z.

83. The system is indeterminate.

84. The system is inconsistent since if for certain numerical values of the unknowns all the equations of the system became equalities, then by subtracting the first equality from the sum of the other two, we would again obtain an equality. But we obtain $0 = 4$.

85. The system is inconsistent.

86. For $a^3 \neq 27$ the system is evendetermined, and for $a^3 = 27$ it is inconsistent.

87. For $4a^3 - 45a + 58 \neq 0$ the system is evendetermined, and for $4a^3 - 45a + 58 = 0$ it is inconsistent.

88. For $ab \neq 15$ the system is evendetermined, for $a = 3$, $b = 5$ it is indeterminate, and for $ab = 15$ but $a \neq 3$, $b \neq 5$ it is inconsistent.

89. For $ab \neq 12$ the system is evendetermined, for $a = 3$, $b = 4$ it is indeterminate, and for $ab = 12$ but $a \neq 3$, $b \neq 4$ it is inconsistent.

99. *Hint.* Consider a determinant in which the first two rows are not proportional (in particular none of the rows should contain zeros only), and the third row is equal to the sum of the first two, that is, each element is equal to the sum of the corresponding elements of the first two rows.

100. 0. **101.** 0. **102.** 0. **103.** 0. **104.** 0. **105.** 0. **106.** 0. **107.** 0. **108.** 0.

109. 0. Two points (x_1, y_1) and (x_2, y_2) of the plane lie on a single straight line with the point dividing the segment between them in the ratio λ.

110. 0. *Hint.* Add the second and third rows to the first and take advantage of the Vieta formula.

120. *Hint.* Add to the third column of the determinant of the left-hand side of the equation the second column multiplied by $a + b + c$ and subtract the first multiplied by $ab + bc + ca$.

123. 5. **124.** 8. **125.** 13. **126.** 18. **127.** $\frac{n(n-1)}{2}$. **128.** $\frac{n(n+1)}{2}$.

129. $\frac{3n(n-1)}{2}$ inversions. The permutation is even for n equal to $4k$, $4k + 1$ and is odd for n equal to $4k + 2$, $4k + 3$, where k is any nonnegative integer.

130. $\frac{3n(n+1)}{2}$ inversions. The permutation is even for $n = 4k$,

$4k + 3$ and odd for $n = 4k + 1$, $4k + 2$, where k is any nonnegative integer.

131. $\frac{n(3n+1)}{2}$ inversions. The permutation is even for $n = 4k$, $4k + 1$ and odd for $n = 4k + 2$, $4k + 3$, where k is any nonnegative integer.

132. $\frac{n(3n-1)}{2}$ inversions. The permutation is even for $n = 4k$, $4k + 3$ and odd for $n = 4k + 1$, $4k + 2$, where k is any nonnegative integer.

133. $3n(n - 1)$ inversions. The permutation is even for arbitrary n.

134. $n(3n - 2)$ inversions. The parity of the permutation coincides with the parity of n.

135. $n(5n + 1)$ inversions. The permutation is even for arbitrary n.

136. In the permutation $n, n - 1, n - 2, \ldots, 3, 2, 1$. The number of inversions in it is equal to $C_n^2 = \frac{n(n-1)}{2}$. **137.** $k - 1$.

138. $n - k$. **139.** C_n^2.

140. For $n = 4k$, $4k + 1$ it is the same, and for $n = 4k + 2$, $4k + 3$ it is the opposite. Here, k is any nonnegative integer.

141. *Solution.* We take any two elements a_i, a_j in the given permutation $(i < j)$.

If in the given permutation these elements constitute an order, then in the original arrangement a_i comes before a_j, and the indices i, j form an order. But if in the given permutation the elements a_i, a_j form an inversion, then in the original arrangement, a_j comes before a_i, and so their indices j, i also form an inversion.

For this reason, the inversions of the given permutation are in a one-to-one correspondence with the inversions of the permutation of the indices of the elements for a normal arrangement of these elements, and hence the number of both kinds of inversions is the same.

142. *Hint.* In the permutation $a_1, a_2, \ldots, a_n$, transfer the element b_1 to the first position, in the resulting permutation transfer b_2 to the second position, and so forth.

143. For example, $2, 3, 4, \ldots, n, 1$ or $n, 1, 2, \ldots, n - 1$. *Hint.* In the proof, utilize the fact that one transposition can reduce the number of elements that stand in the permutation to the right (or to the left) of their places in the normal arrangement by not more than unity.

144. *Hint.* In the permutation $a_1, a_2, \ldots, a_n$ carry the element b_1 to the first place by means of adjacent transpositions, then, via adjacent transpositions, carry element b_2 to the second place in the resulting permutation, and so on.

145. $C_n^2 - k$. **146.** $\frac{1}{2} n! C_n^2$. *Hint.* Make use of the preceding problem.

147. *Hint.* Use adjacent transpositions to carry 1 to the first place, then 2 to the second place, and so on. Take into account that one adjacent transposition changes the number of inversions by unity.

148. *Hint.* Consider a series of permutations beginning with the permutation 1, 2, . . ., n obtained by the following series of transpositions: first carry unity to the last position, interchanging it with each number on the right, then, in the same manner, transfer the two to the next-to-last position, and so on, until we arrive at the permutation n, $n-1$, . . ., 2, 1.

The assertion may also be proved by induction on the number k.

149. *Solution.* To derive the recurrence relation, note that if in a permutation with k inversions the number $n+1$ is in the last position, then all k inversions are formed by the numbers 1, 2, . . ., n, and there will be (n, k) such permutations; if $n+1$ resides in the next-to-last position, then it forms one inversion, and the numbers 1, 2, . . ., n form $k-1$ inversions, and there will be $(n, k-1)$ such permutations, and so forth; finally, if $n+1$ lies in the first position, then it forms n inversions (this is possible only if $k \geqslant n$), and the numbers 1, 2, . . ., n form $k-n$ inversions, and there will be $(n, k-n)$ such permutations.

By arranging the numbers (n, k) in an array made up of rows with the given n and of columns with the given k, we see, from the recurrence relation, that each number of the $(n+1)$th row is equal to the sum $n+1$ of the numbers of the preceding row, reckoning from the left of the number standing above the desired number (including numbers equal to zero). If, for the sake of convenience of reckoning the positions, we write out the zero values of (n, j) as well when $j > C_n^2$ and if we take into account that $(1, 0) = 1$, $(1, j) = 0$ for $j \geqslant 1$, then we obtain the following array of values (n, k):

n \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0
4	1	3	5	6	5	3	1	0	0	0	0	0	0	0	0	0
5	1	4	9	15	20	22	20	15	9	4	1	0	0	0	0	0
6	1	5	14	29	49	71	90	101	101	90	71	49	29	14	5	1

For example, the number of permutations of six elements with seven or eight inversions is equal to 101.

150. *Hint.* In all permutations, replace the given arrangement by the inverse.

151. (1 4 2) (3 5). The decrement is equal to 3. The substitution is odd.

152. (1 6 3) (2 5) (4). The substitution is odd.

153. (1 8 2) (3) (4 6 7) (5). The substitution is even.

154. (1 5) (2 8 6 4) (3 9 7). The substitution is even.

155. (1 2 3 4 5 6 7 8 9). The substitution is even.

156. (1 4) (2 5) (3 6). The substitution is odd.

157. (1 2) (3 4) . . . $(2n-1, 2n)$. The decrement is equal to n. The parity of the substitution coincides with that of the number n.

158. (1 3) (2) (4 6) (5) . . . $(3n-2, 3n)$ $(3n-1)$. The decrement is equal to n. The parity of the substitution coincides with that of the number n.

159. $(1, 3, 5, \ldots, 2n-1)$ $(2, 4, 6, \ldots, 2n)$. The decrement is equal to $2n-2$. The substitution is even.

160. $(1, 2, 3)$ $(4, 5, 6)$ . . . $(3n-2, 3n-1, 3n)$. The decrement is equal to $2n$. The substitution is even.

161. $(1, 4, 7, \ldots, 3n-2)$ $(2, 5, 8, \ldots, 3n-1)$ $(3, 6, 9, \ldots$ $\ldots, 3n)$. The decrement is equal to $3n-3$. The parity of the substitution is opposite that of the number n.

162. $(1, k+1, 2k+1, \ldots, nk-k+1)$ $(2, k+2, 2k+2, \ldots$ $\ldots, nk-k+2)$ . . . $(k, 2k, 3k, \ldots, nk)$. The decrement is equal to $nk-k$. The substitution is even for even k and for odd k and n, and is odd for odd k and for even n.

163. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$.

164. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$.

165. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 6 & 3 & 2 & 5 & 8 & 9 \end{pmatrix}$.

166. $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ 2 & 1 & 4 & 3 & \ldots & 2n & 2n-1 \end{pmatrix}$.

167. $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ 2 & 3 & 4 & 5 & \ldots & 2n & 1 \end{pmatrix}$.

168. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 3n-2 & 3n-1 & 3n \\ 3 & 1 & 2 & 6 & 4 & 5 & \ldots & 3n & 3n-2 & 3n-1 \end{pmatrix}$.

169. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$.

170. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$.

171. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

172. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.

173. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$.

176. *A. Hint.* Make use of the preceding problem.

177. The identical substitution E. 178. $X = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$.

181. *Hint.* Arrange the numbers of the first row of the substitution in increasing order and then pass from the identical substitution to the given substitution via a series of transpositions in the second row.

182. *Hint.* To prove the existence of the decomposition into transpositions of the number equal to the decrement, multiply the substitution by the transposition of numbers that appear in one cycle and make use of problem 180. To prove the minimality of the number of transpositions, note that when multiplying by one transposition, the decrement cannot be increased by more than unity.

183. *Hint.* If $P = P_1P_2 \ldots P_s$ is any decomposition of the substitution P into transpositions, then make use of the equality

$$P = \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ a_1, & a_2, & \ldots, & a_n \end{pmatrix} \cdot P_1P_2 \ldots P_s$$

and of problems 179 and 182.

184. *Solution.* If X is a substitution permutable with S, then $SX = XS$, whence $X^{-1}SX = S$. Decompose S into cycles: $S = (1, 2)(3, 4) = X^{-1}(1, 2)XX^{-1}(3, 4)X$. A straightforward computation convinces us that $X^{-1}(1, 2)X$ is again a cycle of length 2 obtained from the cycle (1, 2) by replacing the numbers 1 and 2 by those which correspond to them in the substitution X. This also holds true for the cycle (3,4). Thus the substitution X must carry cycles from S into cycles of the same length, and since the decomposition of S into cycles is unique, the cycles either pass into themselves or one into another. Since each cycle of length two can be written in two ways, $(1, 2) = (2, 1)$, $(3, 4) = (4, 3)$, it follows that all substitutions that are permutable with S are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

185. The desired substitutions are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}.$$

186. *Hint.* Demonstrate that none of the numbers $1, 2, \ldots, m-1$ can pass into zero and distinct numbers pass into distinct numbers.

187. $\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 \\ 5, & 1, & 6, & 2, & 7, & 3, & 8, & 4 \end{pmatrix}$.

188. With the minus sign. **189.** With the plus sign. **190.** It is not a term of a determinant. **191.** With the minus sign. **192.** It is not a term of a determinant. **193.** With the sign $(-1)^{n-1}$. **194.** With the sign $(-1)^n$.

195. With the sign $(-1)^{\frac{(n-1)(n-2)}{2}}$. **196.** With the sign $(-1)^{3n} = (-1)^n$.

197. $i = 5$, $k = 1$. **198.** $i = 6$, $k = 2$. **199.** $a_{11}a_{24}a_{32}a_{43} + a_{13}a_{21}a_{32}a_{44} + a_{14}a_{23}a_{32}a_{41}$. **200.** $10x^4 - 5x^3$. **201.** With the plus sign. **202.** With the sign $(-1)^{C_n^2}$. **203.** $a_{11}a_{22}a_{33} \ldots a_{nn}$. **204.** $(-1)^{\frac{n(n-1)}{2}} \cdot a_{1n}a_{2,n-1} \ldots a_{n1}$. **205.** 0. **207.** The roots are the numbers $a_1, a_2, a_3, \ldots, a_n$. *Hint.* Use the assertion that a polynomial of degree n cannot have more than n

distinct roots. 208. $x = 0, 1, 2, \ldots, n-1$. 209. $a_{n-k+1, n-i+1}$. 210. $a_{n-i+1, n-k+1}$. 211. If n is even, the number of elements at even and odd sites is the same and is equal to $\frac{1}{2}n^2$. If n is odd, then the number of elements at even sites is equal to $\frac{1}{2}(n^2+1)$ and at odd sites is equal to $\frac{1}{2}(n^2-1)$.

212. The determinant is multiplied by $(-1)^{n-1}$. 213. The determinant is multiplied by $(-1)^{\frac{n(n-1)}{2}}$. 214. The determinant remains unchanged. 215. The determinant does not change. *Hint.* The given transformation may be replaced by two symmetries about the horizontal and vertical middle lines and by symmetry about the principal diagonal.

216. *Hint.* Transpose the determinant.

217. *Hint.* Transpose the determinant.

218. $n = 4m$, where m is an integer. 219. $n = 4m + 2$, where m is an integer.

221. The determinant is multiplied by $(-1)^n$.

222. The determinant is not changed. *Hint.* Consider the general term of the determinant.

223. *Hint.* Consider the sum of the indices of all elements that appear in the general term of the determinant.

224. *Hint.* Make use of the preceding problem.

229. The determinant will vanish.

230. The determinant will vanish if it is of even order and will be doubled if it is of odd order. *Hint.* Decompose into a sum of determinants in terms of each column.

231. The determinant is multiplied by $(-1)^{C_n^2}$. 232. The determinant is equal to zero. 233. The number of such determinants is equal to $n!$ Their sum is equal to zero.

234. 0. 236. $8a + 15b + 12c - 19d$. 237. $2a - 8b + c + 5d$.

238. $abcd$. 239. $abcd$. 240. $xyzuv$. 243. 0.

244. *Hint.* Multiply the second column of the determinant in the left-hand member of the equation by yz, the third column by xz, and the fourth by xy.

245. *Hint.* Use the Vieta formulas to transform the nth column.

246. *Hint.* Use the Vieta formulas to transform the nth column and carry it to the $(i+1)$th site.

247. *Hint.* Expand with respect to the first column.

253. *Hint.* Expand in terms of the third row.

256. *Hint.* Reduce to the preceding problem.

257. -8. 258. -3. 259. -9. 260. 18. 261. 18. 262. 4. 263. 90. 264. 27. 265. 17. 266. -6. 267. -10. 268. 100. 269. 150. 270. 52. 271. 5. 272. 10. 273. 1. 274. 100. 275. 1.

276. 1/35. *Hint.* Reduce the elements of each row to a common denominator and take it outside the sign of the determinant.

277. 1. 278. $9\sqrt{10}(\sqrt{3}-\sqrt{2})$. 279. $n!$ 280. $n(-1)^{\frac{n(n-1)}{2}}$. 281. $x_1(x_2-a_{12})(x_3-a_{23})\ldots(x_n-a_{n-1,n})$.

282. $(-1)^{\frac{n(n-1)}{2}} b_1 b_2 \ldots b_n$. 283. $2n+1$.

284. $(a_0+a_1+a_2+\ldots+a_n)\,x^n$.

285. $x_1x_2\ldots x_n\left(\frac{a_1}{x_1}+\frac{a_2}{x_2}+\ldots+\frac{a_n}{x_n}\right)$. *Hint.* Take x_i outside the sign of the determinant from the ith column and then to each column add all the succeeding columns.

286. 1. 287. $n\,(-1)^{n-1}$. 288. $(-1)^{n-1}\,(n-1)\,2^{n-2}$. *Hint.* From each row subtract the preceding one, and then add the last column to the others.

289. $(x-1)\,(x-2)\ldots(x-n+1)$.

290. $(-1)^n\,(x-1)\,(x-2)\ldots(x-n)$.

291. $a_0\,(x-a_1)\,(x-a_2)\ldots(x-a_n)$.

292. $(x-a-b-c)(x-a+b+c)(x+a-b+c)(x+a+b-c)$.

293. $(x^2-1)\,(x^2-4)$.

294. x^2z^2. *Hint.* Interchanging the first two rows and the first two columns, prove that the determinant does not change when $-x$ is substituted for x. Verify that for $x=0$ the determinant vanishes and then prove that it is divisible by x^2. Carry through the same reasoning for z.

295. $a_1b_n\prod_{i=1}^{n-1}(a_{i+1}b_i-a_ib_{i+1})$. *Hint.* Obtain the relation

$$D_n=\frac{b_n}{b_{n-1}}\,(a_nb_{n-1}-a_{n-1}b_n)\,D_{n-1}.$$

296. $a_0x_1x_2x_3\ldots x_n+a_1y_1x_2x_3\ldots x_n+a_2y_1y_2x_3\ldots x_n+\ldots$ $\ldots+a_ny_1y_2y_3\ldots y_n$. *Hint.* Obtain the relation $D_{n+1}=x_nD_n+a_ny_1y_2\ldots y_n$. The determinant may be computed differently by expanding in terms of the first row.

297. $-a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$. 298. $a_1a_2\ldots a_n-$ $-a_1a_2\ldots a_{n-1}+a_1a_2\ldots a_{n-2}-\ldots+(-1)^{n-1}a_1+(-1)^n$. 299. $n+1$.

300. $2^{n+1}-1$. 301. $\frac{5^{n+1}-2^{n+1}}{3}$. 302. $9-2^{n+1}$. 303. $5\cdot2^{n-1}-4\cdot3^{n-1}$.

304. $\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$.

305. $x^n+(a_1+a_2+\ldots+a_n)\,x^{n-1}$. *Hint.* Represent the elements outside the principal diagonal in the form $a_i=0+a_i$.

306. $(x_1-a_1)\,(x_2-a_2)\ldots(x_n-a_n)\left(1+\frac{a_1}{x_1-a_1}+\frac{a_2}{x_2-a_2}+\ldots\right.$ $\left.\ldots+\frac{a_n}{x_n-a_n}\right)$. *Hint.* Set $x_i=(x_i-a_i)+a_i$.

307. $(a_1-x_1)\,(a_2-x_2)\ldots(a_n-x_n)-a_1a_2\ldots a_n$. *Hint.* In the upper left-hand corner set $0=1-1$ and express the determinant as a sum of two determinants relative to the first row.

308. $(x_1-a_1b_1)(x_2-a_2b_2)\dots(x_n-a_nb_n)\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\dots\right.$
$\left.\dots+\frac{a_nb_n}{x_n}\right)$.

309. $(n-1)!$ 310. $b_1b_2\dots b_n$. 311. $(-1)^{\frac{n^2-n+2}{2}}\,2(n-2)!$
312. $(-1)^{n-1}n!$ 313. $x^n+(-1)^{n+1}y^n$. 314. 0.
315. $(-1)^{\frac{n(n+1)}{2}}(n+1)^{n-1}$. 316. $(-1)^{n-1}(n-1)$.
317. $(2n-1)(n-1)^{n-1}$. 318. $[a+(n-1)b](a-b)^{n-1}$. 319. 1.
320. 1. 321. $(x-a_1)(x-a_2)\dots(x-a_n)$.
322. $a_0x^n+a_1x^{n-1}+\dots+a_n$. 323. $\frac{x^{n+1}-1}{(x-1)^2}-\frac{n+1}{x-1}$.
324. $\frac{nx^n}{x-1}-\frac{x^n-1}{(x-1)^2}$. 325. $\prod_{k=1}^{n}(1-a_kx)$.
326. $(-1)^{\frac{n(n+1)}{2}}[a_0-a_1+a_2-\dots+(-1)^n a_n]$.
327. $(-1)^{n-1}(n-1)x^{n-2}$. 328. $1!\,2!\,3!\dots n!=1^n2^{n-1}3^{n-2}\dots n$.
329. $\prod_{k=1}^{n}k!$ 330. $\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.
331. $\prod_{1\leqslant i<k\leqslant n+1}(a_i-a_k)$.
332. $2^{\frac{n(n-1)}{2}}\prod_{n\geqslant i>k\geqslant 1}\cos\frac{\varphi_i+\varphi_k}{2}\sin\frac{\varphi_i-\varphi_k}{2}$.
333. $2^{\frac{n(n-1)}{2}}\prod_{1\leqslant i<k\leqslant n}\sin\frac{\varphi_i+\varphi_k}{2}\sin\frac{\varphi_i-\varphi_k}{2}$.
334. $\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$. 335. $2^{\frac{n(n-1)}{2}}a_{1,0}a_{2,0}\dots a_{n-1,0}\times$
$\times\prod_{1\leqslant i<k\leqslant n}\sin\frac{\varphi_i+\varphi_k}{2}\sin\frac{\varphi_i-\varphi_k}{2}$.
336. $\frac{1}{1!\,2!\,3!\dots(n-1)!}\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$. 337. $(-1)^n\,1!\,2!\dots n!$
338. $\prod_{i=1}^{n}\frac{x_i}{x_i-1}\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$. 339. $1!\,3!\,5!\dots(2n-1)!$

340. $\prod_{1\leqslant i<k\leqslant n+1}(a_ib_k-a_kb_i)$. 341. $\prod_{1\leqslant i<k\leqslant n}\sin(\alpha_i-\alpha_k)$.

342. $a_1a_2\ldots a_n\prod_{1\leqslant i<k\leqslant n+1}(x_iy_k-x_ky_i)$, where a_i is the coefficient of x^i in the polynomial $f_i(x, y)$.

343. $(-1)^{n-1}\prod_{i=1}^{n}x_i\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)\left[\sum_{i=1}^{n}\frac{a_i}{x_if'(x_i)}\right]$, where $f(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$. *Hint.* Expand the determinant in terms of the first row.

344. $(x_1+x_2+\ldots+x_n)\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$. *Hint.* Compute the determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \ldots & x_n^n \\ 1 & z & z^2 & \ldots & z^n \end{vmatrix}$$

in two ways: by expanding in terms of the last row and as a Vandermonde determinant. In both expressions, equate the coefficients of z^{n-1}.

345. $x_1x_2\ldots x_n\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

346. $\left(\sum x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_{n-s}}\right)\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$, where the sum is taken over all combinations of $n-s$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n-s}$ from the numbers 1, 2, ..., n.

347. $[x_1x_2\ldots x_n-(x_1-1)(x_2-1)\ldots(x_n-1)]\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$. *Hint.* Express the ith element of the first column as $1=x_i-(x_i-1)$ and represent the determinant as a difference of two determinants.

348. $[2x_1x_2\ldots x_n-(x_1-1)(x_2-1)\ldots(x_n-1)]\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$. *Hint.* Adjoin the first row 1, 0, 0, ..., 0 and first column made up of units, subtract the first column from the others, express the unit in the upper left-hand corner as $2-1$, and represent the determinant as a difference of two determinants with respect to the first row.

349. $2^{(n-1)^2}\prod_{1\leqslant i<k\leqslant n}\sin\frac{\varphi_i+\varphi_k}{2}\sin\frac{\varphi_i-\varphi_k}{2}$. *Hint.* Take advantage of the fact that $\cos k\varphi$ can be expressed in terms of $\cos\varphi$ as a polynomial with leading term $2^{k-1}\cos^k\varphi$ (this can be derived from the de Moivre formula and the equation $1+C_k^2+C_k^4+\ldots=2^{k-1}$).

350. $2^{n(n-1)} \sin\varphi_1 \sin\varphi_2 \ldots \sin\varphi_n \prod_{1\leqslant i<k\leqslant n} \left(\sin\frac{\varphi_i+\varphi_k}{2} \sin\frac{\varphi_i-\varphi_k}{2}\right)$.

Hint. Prove that $\sin k\varphi$ can be represented as a product of $\sin\varphi$ by the polynomial in $\cos\varphi$ with leading term $2^{k-1}\cos^{k-1}\varphi$.

351. $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$. *Hint.* Take advantage of the method of isolating linear factors.

352. $(a+b+c+d+e+f+g+h)(a+b+c+d-e-f-g-h)(a+b-c-d+e+f-g-h)(a+b-c-d-e-f+g+h)(a-b+c-d+e-f+g-h)(a-b+c-d-e+f-g+h)(a-b-c+d+e-f-g+h)(a-b-c+d-e+f+g-h)$.

353. $(x+a_1+a_2+\ldots+a_n)(x-a_1)(x-a_2)\ldots(x-a_n)$. *Hint.* Apply isolation of linear factors, or from each row subtract the preceding one, and then add to each column all the succeeding columns.

354. 0 for $n>2$; $D_1 = a_1 - b_1$; $D_2 = (a_1-a_2)(b_1-b_2)$.

355. 0 for $n>2$; $D_1 = 1 + x_1y_1$; $D_2 = (x_1-x_2)(y_1-y_2)$.

356. 0 for $n>1$. *Hint.* Expand into a sum of determinants with respect to each column.

357. $1+\sum_{i=1}^{n}(a_i+b_i)+\sum_{1\leqslant i<k\leqslant n}(a_i-a_k)(b_k-b_i)$.

358. $(-1)^n\left[1-n-\sum_{i=1}^{n}x_iy_i+\sum_{1\leqslant i<k\leqslant n}(x_i-x_k)(y_i-y_k)\right]$. *Hint.* Apply the result of problem 355.

359. $x^n+x^{n-1}\sum_{i=1}^{n}(a_i-b_i)+x^{n-2}\sum_{1\leqslant i<k\leqslant n}(a_i-a_k)(b_i-b_k)$. *Hint.* Expand the determinant into a sum of two determinants with respect to each column and then apply the result of problem 354.

360. $x_1x_2\ldots x_n\left(1+\sum_{i=1}^{n}\frac{a_ib_i}{x_i}\right)$. **361.** $(a^2-b^2)^n$. *Hint.* Derive the recurrence relation $D_{2n} = (a^2-b^2)D_{2n-2}$.

362. $\prod_{i=1}^{n}(a_ia_{2n+1-i}-b_ib_{2n+1-i})$. **363.** $\frac{n+1}{x^n}$.

364. 0 if n yields a remainder of 2 or 5 when divided by 6; 1 if n is divisible by 6 or yields a remainder of one; -1 if n yields a remainder of 3 or 4 when divided by 6. The answer can also be written down as

$$D_n = \frac{C_{n+1}^1 - 3C_{n+1}^3 + 9C_{n+1}^5 - 27C_{n+1}^7 + \ldots}{2^n}.$$

Hint. Apply the method of recurrence relations discussed in the introduction to this section.

365. *Hint.* Obtain the recurrence relation $D_n = D_{n-1} + D_{n-2}$.

366. $5^{n+1}-4^{n+1}$. **367.** $\frac{i^n[1+(-1)^n]}{2}$, where $i=\sqrt{-1}$; (that is if n is odd, then $D_n=0$; if n is even, then $D_n=(-1)^{\frac{n}{2}}$.

368. $\frac{1}{2}[1+(-1)^n]$. **369.** $D_n=\frac{1}{2^n}[C_{n+1}^1 a^n+C_{n+1}^3 a^{n-2}(a^2-4)+$

$+C_{n+1}^5 a^{n-4}(a^2-4)^2+C_{n+1}^7 a^{n-6}(a^2-4)^3+\ldots]=a^n-C_{n-1}^1 a^{n-2}+$

$+C_{n-2}^2 a^{n-4}-\ldots+(-1)^k C_{n-k}^k a^{n-2k}+\ldots$. *Hint.* The first expression is obtained directly by the method of recurrence relations, the second can readily be proved by the method of induction by using the relation $C_k^p+C_n^{p+1}=C_{n+1}^{p+1}$.

370. $D_n=\frac{1}{2^n}[C_{n+1}^1 a^n+C_{n+1}^3 a^{n-2}(a^2+4)+\ldots+C_{n+1}^{2k+1} a^{n-2k}(a^2+4)^k+$

$+\ldots]=a^n+C_{n-1}^1 a^{n-2}+C_{n-2}^2 a^{n-4}+\ldots+C_{n-k}^k a^{n-2k}+\ldots$.

371. $2\cos n\alpha=(2\cos\alpha)^n-n(2\cos\alpha)^{n-2}+\frac{n(n-3)}{2!}(2\cos\alpha)^{n-4}-$

$-\frac{n(n-4)(n-5)}{3!}\cdot(2\cos\alpha)^{n-6}+\ldots=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{n}{n-k}C_{n-k}^k(2\cos\alpha)^{n-2k}$,

where $\left[\frac{n}{2}\right]$ denotes the integral part of the number $\frac{n}{2}$. *Hint.* Prove by induction on n that $\cos n\alpha$ is equal to the given determinant.

And if D_n is the determinant of the given problem, and D_n^0 is the determinant of problem 369 with $2\cos\alpha$ substituted for a, then $D_n=D_n^0-\cos\alpha\, D_{n-1}^0$. The fact that the coefficients in the resulting expression of $\cos n\alpha$ in terms of $\cos\alpha$ are integral follows from the easily verifiable equality $\frac{n}{n-k}C_{n-k}^k=C_{n-k}^k-C_{n-k-1}^{k-1}$ and from the fact that all terms, except the last one, contain the factor 2, and the last term fails to involve $2\cos\alpha$ only for even n, but then $k=\frac{n}{2}$ and this term is equal to 2.

372. $\sin n\alpha=\sin\alpha[(2\cos\alpha)^{n-1}-C_{n-2}^1(2\cos\alpha)^{n-3}+C_{n-3}^2(2\cos\alpha)^{n-5}-$

$-C_{n-4}^3(2\cos\alpha)^{n-7}+\ldots]=\sin\alpha\sum_{k=0}^{\left[\frac{n-1}{2}\right]}C_{n-k-1}^k(2\cos\alpha)^{n-2k-1}$, where

$\left[\frac{n-1}{2}\right]$ is the integral part of the number $\frac{n-1}{2}$.

373. *Hint.* Apply the method of recurrence relations.

374. $1+x^2+x^4+\ldots+x^{2n}$. 375. $(-1)^{\frac{n(n-1)}{2}}\frac{n+1}{2}n^{n-1}$.

376. $(-nx)^{n-1}\left[a+\frac{(n-1)x}{2}\right]$. *Hint.* From each row subtract the next row, add all columns to the following column, add to the next-to-last row all preceding rows, and then add this row to all preceding rows.

377. $(1-x^n)^{n-1}$.

378. These circulants differ solely in the factor $(-1)^{\frac{(n-2)(n-1)}{2}}$.

379. 1. *Hint.* Using the equality $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$. subtract from each column the preceding column, and then subtract from each row the preceding row.

380. 1. *Hint.* Take advantage of the hint of problem 379.

381. 1. *Hint.* From each row subtract the preceding row.

382. 1. *Hint.* From each row, beginning with the second onwards, subtract the preceding row, and then from each row, beginning with the third, subtract the preceding row, etc.

383. $(-1)^{\frac{n(n+1)}{2}}$. *Hint.* From each column, beginning with the second, subtract the preceding column, and then from each column, beginning with the third, subtract the preceding column, and so on. Take the resulting determinant and do the same with respect to the rows.

384. 1. *Hint.* Take advantage of the hint referring to problem 382.

385. $\dfrac{\binom{m+n}{n+1}\binom{m+n-1}{n+1}\cdots\binom{m+n-p+1}{n+1}}{\binom{p+n}{n+1}\binom{p+n-1}{n+1}\cdots\binom{n+1}{n+1}}$. *Hint.* Using the relation $\binom{n}{k}=\frac{n}{k}\cdot\binom{n-1}{k-1}$, take m outside the sign of the determinant out of the first row, $m+1$ out of the second row, and so forth, and $m+n$ out of the last row, then take $\frac{1}{p}$ out of the first column, $\frac{1}{p+1}$ out of the second column, and so forth, and $\frac{1}{p+n}$ out of the last column. Take the resulting determinant and perform similar operations and so forth until you arrive at a determinant of the same type as in the preceding problem.

386. n. *Hint.* From each row subtract the preceding row, expand in terms of elements of the first column, take the resulting determinant and from the first column subtract the second, add the third, subtract the fourth, and so on. Represent the determinant as a sum of two determinants and show that $D_n = D_{n-1} + 1$.

387. n. *Hint.* From each column subtract the preceding one, then from each row subtract the preceding row. Represent element 2 in the

upper left-hand corner as $1 + 1$ and obtain the relation $D_{n-1} = D_{n-2} + D'_{n-1}$, where D'_{n-1} is a determinant of the same type as in problem 379 but of order $n - 1$.

388. $(x-1)^n$. *Hint.* From each row subtract the preceding row and show that $D_{n+1} = (x-1)\,D_n$.

389. $1!\ 2!\ 3!\ \ldots\ (n-1)!\ (x-1)^n$. *Hint.* Reduce this problem to the preceding problem.

390. $x_n - \binom{n}{1} x_{n-1} + \binom{n}{2} x_{n-2} - \binom{n}{3} x_{n-3} + \ldots + (-1)^n x_0$. *Hint.* From each row subtract the preceding row, show that $D_{n+1}(x_0, x_1, \ldots, x_n) = D_n(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1})$ and apply the method of mathematical induction.

391. $(-1)^{n-1} xy \dfrac{x^{n-1} - y^{n-1}}{x-y}$. *Hint.* In the lower right-hand corner set $0 = x - x$, decompose into two determinants and either apply the method of recurrence relations or find D_n from the following two equalities:

$$D_n = -xD_{n-1} + x(-y)^{n-1}, \quad D_n = -yD_{n-1} + y(-x)^{n-1}.$$

392. $\dfrac{x(a-y)^n - y(a-x)^n}{x-y}$.

393. $\dfrac{xf(y) - yf(x)}{x-y}$, where $f(z) = (a_1 - z)(a_2 - z)\ldots(a_n - z)$.

394. $\dfrac{f(x) - f(y)}{x-y}$, where $f(z) = (a_1 - z)(a_2 - z)\ldots(a_n - z)$.

395. $\alpha^n + \beta^n$. 396. $a^n + a^{n-1} + \ldots + a + 1$.

397. $n!\,(a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n)$.

398. $\dfrac{\tan^n \dfrac{\alpha}{2} + \cot^n \dfrac{\alpha}{2}}{2}$.

399. $\prod_{k=1}^{n} (x + n - 2k + 1)$ or $(x^2 - 1^2)(x^2 - 3^2)\ldots[x^2 - (n-1)^2]$, if n is even and $x(x^2 - 2^2)(x^2 - 4^2)\ldots[x^2 - (n-1)^2]$ if n is odd. *Hint.* To each row add all the following rows, from each column subtract the preceding column and show that if $D_n(x)$ is the given determinant, then $D_n(x) = (x + n - 1)D_{n-1}(x - 1)$.

400. 0 if $n > 2$, $D_1 = a^p - x$; $D_2 = xa^p(a^2 - 1)(1 - a)$.

401. $(-1)^n \left[x^n - x^{n-1} \dfrac{a^{2n} - 1}{a^2 - 1} \right]$.

402. $a_1 a_2 \ldots a_n \left(a_0 - \dfrac{1}{a_1} - \dfrac{1}{a_2} - \ldots - \dfrac{1}{a_n} \right)$.

403. $abc_1c_2\ldots c_n \left(\dfrac{c_0}{ab} - \dfrac{1}{c_1} - \dfrac{1}{c_2} - \ldots - \dfrac{1}{c_n} \right)$.

404. $a(a+b)(a+2b)\ldots[a+(n-1)b] \left(\dfrac{1}{a} + \dfrac{1}{a+b} + \dfrac{1}{a+2b} + \right.$ $\left. + \ldots + \dfrac{1}{a+(n-1)b} \right)$. *Hint.* From each row, beginning with the second, subtract the following row, from the first row subtract the

last row and obtain a determinant of the same type as that in the preceding problem.

405. $\left[1+\sum_{i=1}^{n}\frac{a_i^2}{x_i^2-2a_ix_i}\right]\prod_{i=1}^{n}(x_i^2-2a_ix_i)$. *Hint.* Use the result of problem 306.

406. $\left[1+\sum_{i=1}^{n}\frac{a_i^2}{x_i^2-2a_ix_i}\right]\prod_{i=1}^{n}(x_i^2-2a_ix_i)$.

407. $1-b_1+b_1b_2-b_1b_2b_3+\ldots+(-1)^nb_1b_2\ldots b_n$. *Hint.* Obtain the relation $D_n=1-b_1D_{n-1}$.

408. $(-1)^{n-1}(b_1a_2a_3\ldots a_n+b_1b_2a_3\ldots a_n+\ldots+b_1b_2\ldots b_{n-1}a_n)$.

409. $(-1)^{n-1}x^{n-2}$. *Hint.* From each row subtract the next row.

410. $(-1)^n[(x-1)^n-x^n]$. *Hint.* From each row subtract the preceding row, set $1=x+(1-x)$ in the lower right-hand corner, and express the determinant as a sum of two determinants.

411. $a_0x^n\prod_{i=1}^{n}(b_i-a_i)$. *Hint.* Multiply the second row by x^{n-1}, the third by x^{n-2}, and so on, and the nth by x. Take x^n out of the first column, x^{n-1} out of the second column, x^{n-2} out of the third column, and so forth, and take x out of the nth column.

412. $n!\left(1+x+\frac{x}{2}+\frac{x}{3}+\ldots+\frac{x}{n}\right)$.

413. $2^{\frac{n(n+1)}{2}}\left[1+x\left(2-\frac{1}{2^n}\right)\right]$.

414. $\frac{1}{n!}\left[1+\frac{n(n+1)}{2}x\right]$. 415. $a^{\frac{n(n+1)}{2}}\left[1+x\frac{a^{n+1}-1}{a^n(a-1)}\right]$.

416. $\dfrac{\prod_{1\leqslant i<k\leqslant n}[(a_i-a_k)(b_i-b_k)]}{\prod_{i,\,k=1}^{n}(a_i+b_k)}$, where the product in the denominator is taken over all i, k, which independently run through all values from 1 to n. *Hint.* In each row, take outside the sign of the determinant the common denominator of the elements of that row. Show that the resulting determinant D' is divisible by all differences of the type a_i-a_k and b_i-b_k $(i\neq k)$. Show that the quotient obtained from the division of D' by $\prod_{1\leqslant i<k\leqslant n}[(a_i-a_k)\times(b_i-b_k)]$ is a constant; to determine this constant, in D' put

$$a_1=-b_1,\ a_2=-b_2,\ \ldots,\ a_n=-b_n.$$

This can be done differently: from each row subtract the first row, and then from each column subtract the first column.

417. $\dfrac{\prod\limits_{1\leqslant i<k\leqslant n}[(x_i-x_k)(a_k-a_i)]}{\prod\limits_{i,\,k=1}^{n}(x_i-a_k)}$. *Hint.* Take advantage of the preceding problem.

418. $\dfrac{[1!\,2!\,3!\ldots(n-1)!]^3}{n!\,(n+1)!\,(n+2)!\ldots(2n-1)!}$. *Hint.* Take advantage of the results of problem 416.

419. $a_0a_1a_2\ldots a_n\left(\dfrac{1}{a_0}+\dfrac{1}{a_1}+\ldots+\dfrac{1}{a_n}\right)$. *Hint.* Use the recurrence relation $D_n=(a_{n-1}+a_n)D_{n-1}-a_{n-1}^2D_{n-2}$ and apply the method of mathematical induction.

420. The continuant $(a_1a_2\ldots a_n)$ is equal to the sum of all possible products of the elements $a_1, a_2, \ldots, a_n$, one of which contains all these elements and the others are obtained from it by deleting one or several pairs of factors with adjacent numbers. Here, the term obtained by deleting all factors (for even n) is taken to be equal to 1;

$$(a_1a_2a_3a_4)=a_1a_2a_3a_4+a_3a_4+a_1a_4+a_1a_2+1,$$

$$(a_1a_2a_3a_4a_5)=a_1a_2a_3a_4a_5+a_3a_4a_5+a_1a_4a_5+a_1a_2a_5+a_1a_3a_5+a_1+a_3+a_5,$$

$$(a_1a_2a_3a_4a_5a_6)=a_1a_2a_3a_4a_5a_6+a_3a_4a_5a_6+a_1a_4a_5a_6+a_1a_2a_5a_6+a_1a_2a_3a_6+a_1a_2a_3a_4+a_5a_6+a_3a_6+a_1a_6+a_3a_4+a_1a_4+a_1a_2+1.$$

Hint. Verify the truth of this law for continuants of 1st and 2nd order and, assuming it to hold for continuants of orders $(n-1)$ and $(n-2)$, prove that it holds true for continuants of the nth order. To do this, derive the recurrence relation

$$(a_1a_2\ldots a_n)=a_n(a_1a_2\ldots a_{n-1})+(a_1a_2\ldots a_{n-2}).$$

421. $(C_n^k)^2$.

424. *Hint.* Show that the number of inversions in both rows of the given substitution is equal to $\alpha_1+\alpha_2+\ldots+\alpha_k+\beta_1+\beta_2+\ldots+\beta_k-2(1+2+\ldots+k)$.

425. 10. 426. 100. 427. 60. 428. 10. 429. -4. 430. -2. 431. 195. 432. 90. 433. 8. 434. 4. 435. 1000. 436. 12. 437. $(x_2-x_1)\sin(\gamma-\beta)+(y_2-y_1)\sin(\alpha-\gamma)+(z_2-z_1)\sin(\beta-\alpha)$.

438. $A^2x_1+B^2x_2+C^2x_3+2BCy_1+2CAy_2+2ABy_3$, where $A=bc'-b'c$, $B=ca'-c'a$, $C=ab'-a'b$.

439. $-(ayz+bxz+cxy)$. 440. $-(aa'+bb'+cc')$. 441. $abc-x(bc+ca+ab)$. 442. $(x_4-x_3)[(x_3-x_2)(x_4-x_2)-2(x_3-x_1)(x_4-x_1)]$.

443. $\prod\limits_{k=1}^{n}(a_{kk}a_{2n-k+1,\,2n-k+1}-a_{k,\,2n-k+1}\,a_{2n-k+1,\,k})$.

444. $(-1)^n\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)^2$.

445. —84. *Hint.* From the second row subtract twice the first and to the third row add twice the fourth.

446. —84. 447. 98. 448. 43. 449. 81. 450. 14. 451. $(-1)^n (nx + 1) x^n$. 452. $b_{1n} b_{2, n-1} \ldots b_{n1} (a_{1n} - c_{1n}) (a_{2n-1} - c_{2n-1}) \ldots (a_{n1} - c_{n1})$. 453. $x^{2n} - x^{2n-2} (a_1 + a_2 + \ldots + a_n)^2$.

454. (a) $D = M_1 M_4$; (b) $D = (-1)^k M_2 M_3$.

455. $D = (-1)^{k + \frac{k(k+1)l}{2}} M_1 M_2 \ldots M_l$. The rule of signs can be stated differently: for even l we take the sign $(-1)^k$ and for odd l the sign $(-1)^{\frac{k(k-1)}{2}}$.

459. $(2^{k+1} - 1)(3^{l+1} - 2^{l+1}) - 4(2^k - 1)(3^l - 2^l)$. *Hint.* Expand in terms of the first k rows and apply the method of recurrence relations.

460. $(a_1 a_2 \ldots a_n) = (a_1 a_2 \ldots a_k)(a_{k+1} a_{k+2} \ldots a_n) + (a_1 a_2 \ldots a_{k-1})(a_{k+2} a_{k+3} \ldots a_n)$. Assuming $n = 2k$, $a_1 = a_2 = \ldots = a_n = 1$ here and denoting by u_n the nth number in the Fibonacci series, we obtain $u_{2k} = u_k^2 + u_{k-1}^2$, that is to say, the sum of the squares of two adjacent numbers in the Fibonacci series is also a number of that series.

462. *Hint.* Consider a determinant of order $2n$ of the matrix obtained from the given matrix by adjoining, below, the same n rows and in the same order.

463. *Hint.* Expanding D in terms of the 1st, 3rd, and 5th rows, show that $D = A\Delta^2$, where A is independent of the elements of Δ. To determine A, set the elements on the principal diagonal of Δ equal to 1 and those outside the principal diagonal of Δ equal to zero.

464. *Hint.* Decompose the determinant in the left-hand member of the equation into a sum of determinants with respect to each row and represent it as

$$\sum_{i,\, j,\, k=0}^{4} a_i b_j c_k D_{ijk} \quad \text{where} \quad D_{ijk} = \begin{vmatrix} \alpha^i & \beta^i & \gamma^i \\ \alpha^j & \beta^j & \gamma^j \\ \alpha^k & \beta^k & \gamma^k \end{vmatrix}.$$

Show that this sum may be taken only over all triplets ijk not containing equal numbers if we expand the 5th-order determinant in the right-hand member of the equation in terms of the first three rows and represent the right-hand member as $\sum a_i b_j c_k C_{ijk}$, where the sum is taken over all triplets of distinct number ijk that run from 0 to 4. Finally, show that $D_{ijk} = C_{ijk}$; to do this, reduce any triplet of numbers ijk to the case $i < j < k$ by a permutation of rows and columns of the determinants and consider all ten possible cases. For example,

$$\Delta_{0,1,3} = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta).$$

But also $C_{0,1,3} = -\begin{vmatrix} p & 0 \\ q & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha) \times (\gamma-\beta)$, since $p = -(\alpha+\beta+\gamma)$.

466. *Solution.* The proof follows the same pattern as in the Laplace theorem. We will show that any term of the product $\varepsilon M_1 M_2 \ldots M_p$ is a term of the determinant D. First let M_1 lie in the first k rows and the first k columns, M_2 in the next l rows and the next l columns, and so forth, and M_p in the last s rows and the last s columns. In that case the substitution (1) is an identical substitution and $\varepsilon = +1$.

We now take the product of any terms of the minors M_1, M_2, ..., M_p in increasing order of the first indices of the elements. It contains one element each from every row and every column, and hence is, by the composition of elements, a term of D. If in the second indices of the elements of the term of the minor M_i there are σ_i inversions, then the sign of that product is $(-1)^{\sigma_1+\ldots+\sigma_p}$. But the indices of the elements of two distinct minors M_i and M_j do not form inversions. Thus, $\sigma_1 + \ldots + \sigma_p$ is the total number of inversions in the second indices of the elements of the product taken, and it will be a term of D in sign as well. Now let the minors M_i be arranged in arbitrary fashion. Let us carry them into the above-considered position on the principal diagonal by the following permutations of rows and columns of D. To begin with, carry the first row of the minor M_1, which has the number α_1, to the first position by interchanging it with all above-lying rows of D. In doing so, we perform $\alpha_1 - 1$ transpositions of rows, that is, just as many as there are inversions formed by the number α_1 in the upper row of the substitution (1) with the numbers that follow it. In the same manner, the row with the number α_2 is carried into the second position via the same number of transpositions of rows as there are inversions formed by α_2 in the upper row of the substitution (1) with the numbers that follow it, and so on. The columns of D are interchanged in the same manner. If there are σ inversions in the first row of the substitution (1) and τ inversions in the second row, then $\varepsilon = (-1)^{\sigma+\tau}$, and altogether we perform $\sigma + \tau$ transpositions of rows and columns of D. We therefore arrive at the new determinant D' for which

$$D = \varepsilon D'. \tag{2}$$

By what has already been proved, any term of the product $M_1 M_2 \ldots M_p$ will be a term of the determinant D' and by virtue of (2) any term of the product $\varepsilon M_1 M_2 \ldots M_p$ will be a term of the determinant D.

All terms of one and the same or of two distinct products $\varepsilon M_1 M_2 \ldots M_p$ differ from one another in the composition of the elements and therefore are distinct terms of the determinant D. It remains to prove that the total number of terms of all such products is equal to $n!$ The number of minors M_1 is equal to C_n^k. If M_1 has already been chosen, then the minors M_2 can only lie in the remaining $n - k$ rows and their number, for each choice of M_1, is equal to C_{n-k}^l. For the chosen M_1 and M_2, the number of minors M_3 is equal to C_{n-k-l}^m and so on; finally, for the chosen $M_1, M_2, \ldots, M_{p-1}$ the number of minors M_p is equal to $C_s^s = 1$. For this reason, the total number

of products of the form $\varepsilon M_1 M_2 \ldots M_p$ is

$$C_n^k \cdot C_{n-k}^l C_{n-k-l}^m \cdots C_s^s$$
$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{(n-k)!}{l!\,(n-k-l)!} \cdot \frac{(n-k-l)!}{m!\,(n-k-l-m)!} \cdots \frac{s!}{s!} = \frac{n!}{k!\,l!\,m!\ldots!\,s!}.$$

But the number of terms of the determinant D in each product $\varepsilon M_1 M_2 \ldots M_p$ is equal to $k!l!m!\ldots!s!$. Hence, the number of terms in all products $\varepsilon M_1 M_2 \ldots M_p$ is equal to

$$\frac{n!}{k!\,l!\,m!\ldots!\,s!}\, k!\,l!\,m!\ldots!\,t! = n!$$

467. We obtain, by multiplying

rows into rows: $\begin{vmatrix} -1 & 2 & -3 \\ -4 & -7 & -13 \\ -3 & -4 & -13 \end{vmatrix}$,

rows into columns: $\begin{vmatrix} 7 & -26 & 13 \\ 12 & -35 & 19 \\ 17 & -52 & 27 \end{vmatrix}$,

columns into rows: $\begin{vmatrix} -3 & 1 & -6 \\ -3 & 1 & -8 \\ 4 & 7 & 1 \end{vmatrix}$.

columns into columns: $\begin{vmatrix} 9 & -35 & 18 \\ 13 & -47 & 24 \\ 12 & -37 & 17 \end{vmatrix}$.

The values of the given determinants are -5 and 10, and the values of all determinants obtained are equal to -50.

468. $(a^2 + b^2 + c^2 + d^2)^2$. **469.** $(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2)^4$.

470. 0 for $n > 2$; $(x_2 - x_1)(y_2 - y_1)$ for $n = 2$. *Hint.* Express

$$\begin{vmatrix} 1 & x_1 & 0 & \ldots & 0 \\ 1 & x_2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & 0 & \ldots & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & y_1 & 0 & \ldots & 0 \\ 1 & y_2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & y_n & 0 & \ldots & 0 \end{vmatrix}$$

as a product of determinants.

471. 0 if $n > 2$, $\sin(\alpha_1 - \alpha_2) \sin(\beta_1 - \beta_2)$ if $n = 2$.

472. 0 if $n > 2$, $\sin^2(\alpha_1 - \alpha_2)$ if $n = 2$.

473. 0 if $n > 2$, $-\sin^2(\alpha_1 - \alpha_2)$ if $n = 2$.

474. $\prod_{n \geqslant i > k \geqslant 1} (a_i - a_k)(b_i - b_k)$.

475. $C_n^1 C_n^2 \ldots C_n^n \prod_{n \geqslant i > k \geqslant 0} (a_k - a_i)(b_i - b_k)$.

476. $(-1)^{\frac{n(n-1)}{2}}[(n-1)!]^n$. *Hint.* Write the element in the ith row and the kth column as $[i+(k-1)]^{n-1}$ and expand via the formula of the power of a binomial or directly take advantage of the result of the preceding problem.

477. $\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)^2$. 478. $\prod_{i=1}^{n}(x-x_i)\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)^2$. *Hint.* Express the following as a product of determinants:

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & x \\ x_1^2 & x_2^2 & \dots & x_n^2 & x^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} & x^{n-1} \\ x_1^n & x_2^n & \dots & x_n^n & x^n \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 1 & \dots & 1 & 0 \\ x_1 & x_2 & \dots & x_n & 0 \\ x_1^2 & x_2^2 & \dots & x_n^2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

The product is set up in terms of rows.

479. *Hint.* Multiply the given determinant by the Vandermonde determinant

$$v=\begin{vmatrix} 1 & \varepsilon_1 & \varepsilon_1^2 & \dots & \varepsilon_1^{n-1} \\ 1 & \varepsilon_2 & \varepsilon_2^2 & \dots & \varepsilon_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon_n & \varepsilon_n^2 & \dots & \varepsilon_n^{n-1} \end{vmatrix}.$$

481. $(1-\alpha^n)^{n-1}$. *Hint.* Make use of the result of problem 479 and the equation $(1-\alpha\varepsilon_1)(1-\alpha\varepsilon_2)\dots(1-\alpha\varepsilon_n)=1-\alpha^n$, where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are nth roots of unity. It is simpler, however, to compute this determinant as a special case of the determinants of problem 325.

483. $(a+b+c+d)(a-b+c-d)(a+bi-c-di)(a-bi-c+di)=a^4-b^4+c^4-d^4-2a^2c^2+2b^2d^2-4a^2bd+4b^2ac-4c^2bd+4d^2ac$.

484. $[1+(-1)^n]^n=\begin{cases} 0 & \text{for odd } n, \\ 2^n & \text{for even } n. \end{cases}$

485. $(-1)^n\dfrac{[(n+1)a^n-1]^n-n^na^{n(n+1)}}{(1-a^n)^2}$.

486. *Hint.* Compute the first determinant by using the result of problem 479.

487. $(-2)^{n-1}(n-2p)$ if n and p are relatively prime; 0 if n and p are not relatively prime. *Hint.* Use the result of problem 479 and the properties of the nth roots of unity, in particular the fact that when p is relatively prime to n, the numbers $\varepsilon_1^p, \varepsilon_2^p, \dots, \varepsilon_n^p$ are again all values of $\sqrt[n]{1}$ and for p not relatively prime to n there is an $\varepsilon_k\neq 1$ for which $\varepsilon_k^p=1$.

488. $[3+(n-p)b](a-b)^{n-1}$ if n and p are relatively prime; 0 if n and p are not relatively prime. *Hint.* Make use of the hint of the preceding problem.

489. $2^{n-2}\left(\cos^n\frac{\pi}{n}-1\right)$. *Hint.* Set $\cos\frac{j\pi}{n}=\frac{\varepsilon_1^j+\varepsilon_1^{-j}}{2}$, where $\varepsilon_1=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$, and make use of the result of problem 479 and also of the fact that for any a we have $\prod_{k=0}^{n-1}(a-\varepsilon_1^{2k})=a^n-1$ and $\varepsilon_1^n=-1$.

490. $$\frac{[\cos\theta-\cos(n+1)\theta]^n-(1-\cos n\theta)^n}{2(1-\cos\theta)}$$
$$=2^{n-2}\sin^{n-2}\frac{n\theta}{2}\left[\sin^n\frac{(n+2)\theta}{2}-\sin^n\frac{n\theta}{2}\right].$$

Hint. Set $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, $\eta=\cos\theta+i\sin\theta$ and make use of the result of problem 479.

491. $(-1)^n 2^{n-2}\sin^{n-2}\frac{nh}{2}\left[\cos^n\left(a+\frac{nh}{2}\right)-\cos^n\left(a+\frac{(n-2)h}{2}\right)\right]$.

492. $(-1)^{n-1}\frac{(n+1)(2n+1)n^{n-2}}{12}[(n+2)^n-n^n]$.

Hint. Make use of the result of problem 479 and of the relations $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ and $1+4\varepsilon+9\varepsilon^2+\ldots+n^2\varepsilon^{n-1}=$ $=-\frac{n^2(1-\varepsilon)+2n}{(1-\varepsilon)^2}$, where ε is the nth root of 1, which root is different from 1. To obtain this last equation, multiply and divide the left-hand member by $1-\varepsilon$.

493. $f(\eta_1)f(\eta_2)\ldots f(\eta_n)$, where $f(x)=a_1+a_2x+a_3x^2+\ldots+a_nx^{n-1}$ and $\eta_1, \eta_2, \ldots, \eta_n$ are all values of the root $\sqrt[n]{-1}$, for example, $\eta_k=\cos\frac{(2k-1)\pi}{n}+i\sin\frac{(2k-1)\pi}{n}$. *Hint.* Multiply the given determinant by the Vandermonde determinant made up of the numbers $\eta_1, \eta_2, \ldots, \eta_n$.

494. $f(\alpha_1)f(\alpha_2)\ldots f(\alpha_n)$, where $f(x)=a_1+a_2x+a_3x^2+\ldots+a_nx^{n-1}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all values of the nth root of z.

495. *Hint.* Denoting the $2n$th roots of 1 by $\varepsilon_k=\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$, $k=0, 1, 2, \ldots, 2n-1$, show that the numbers ε_k with even indices k yield all the nth roots of unity; and $a_1+a_2\varepsilon_k+a_3\varepsilon_k^2+\ldots+a_{2n}\varepsilon_k^{2n-1}=$ $=(a_1+a_{n+1})+(a_2+a_{n+2})\varepsilon_k+(a_3+a_{n+3})\varepsilon_k^2+\ldots+(a_n+a_{2n})\varepsilon_k^{n-1}$ and the number ε_k with odd indices k yield all the nth roots of -1; we have $a_1+a_2\varepsilon_k+a_3\varepsilon_k^2+\ldots+a_{2n}\varepsilon_k^{2n-1}=(a_1-a_{n+1})+(a_2-a_{n+2})\varepsilon_k+$ $+(a_3-a_{n+3})\varepsilon_k^2+\ldots+(a_n-a_{2n})\varepsilon_k^{n-1}$.

496. The product of two numbers, each of which is the sum of the squares of four integers, is itself the sum of the squares of four integers. *Hint.* Square each of the determinants.

497. The product of two numbers, each of which is equal to the value of the form $x^3 + y^3 + z^3 - 3xyz$ for integral values of x, y, z, is itself a number of the same type. *Hint.* Compute the product of the determinants

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \cdot \begin{vmatrix} a' & b' & c' \\ c' & a' & b' \\ b' & c' & a' \end{vmatrix}$$

by multiplying the rows of the first by the columns of the second.

498. *Hint.* In the product of the determinants

$$\begin{vmatrix} a & 1 & b \\ b & 1 & a \\ c & 1 & b \end{vmatrix} \cdot \begin{vmatrix} a' & b' & 1 \\ c' & a' & 1 \\ b' & c' & 1 \end{vmatrix}$$

made up by multiplying columns by columns, multiply the third columns by $s' = a' + b' + c'$ and take the factor s' out of the second row of the determinant. Then subtract the first and second columns from the third column.

499. *Hint.* Write down the determinant D in the form

$$D = \begin{vmatrix} \sum a_{1, k_1} b_{1, k_1} & \sum a_{1, k_2} b_{2, k_2} & \cdots & \sum a_{1, km} b_{m, km} \\ \sum a_{2, k_1} b_{1, k_1} & \sum a_{2, k_2} b_{2, k_2} & \cdots & \sum a_{2, km} b_{m, km} \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{m, k_1} b_{1, k_1} & \sum a_{m, k_2} b_{2, k_2} & \cdots & \sum a_{m, km} b_{m, km} \end{vmatrix},$$

where all sums of the jth column are taken over one and the same index $k_j = 1, 2, \ldots, n$; then expand D into a sum of n^m determinants with respect to columns; in each summand of the jth column take $b_{j, kj}$ outside the sign of the determinant and show that

$$D = \sum_{k_1, k_2, \ldots, km=1}^{n} b_{1, k_1} b_{2, k_2} \cdots b_{m, km} A_{k_1, k_2, \ldots, km}, \qquad (3)$$

where the summation indices vary from one to n independently. Note that $A_{k_1, k_2, \ldots, km} = 0$ if there are equal indices among the indices $k_1, k_2, \ldots, k_m$. From this derive the statement (2) and for $m \leqslant n$ prove that for any indices $i_1, i_2, \ldots, i_m$, where $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant n$, all summands of the sum (3) in which the indices $k_1, k_2, \ldots, k_m$ form any permutations of the numbers $i_1, i_2, \ldots, i_m$ have a sum equal to $A_{i_1, i_2, \ldots, i_m} \cdot B_{i_1, i_2, \ldots, i_m}$, and from this obtain statement (1).

500. *Hint.* Complete the matrices A and B to obtain square matrices with the aid of $m - n$ columns consisting solely of zeros.

501. *Hint.* Apply the theorem of problem 499 to the matrices

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{pmatrix}.$$

503. *Hint.* Take advantage of the identity of the preceding problem.

504. *Hint.* Apply the theorem of problem 499 to the matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \\ \bar{b}_1 & \bar{b}_2 & \dots & \bar{b}_n \end{pmatrix}.$$

505. *Hint.* Take advantage of the identity of the preceding problem.

506. *Hint.* By multiplying D and D' with respect to rows, show that $DD' = D^n$, whence follows (1) for $D \neq 0$. When $D = 0$, consider the case where all elements of D are zero. If $D = 0$, but at least one element $a_{ij} \neq 0$, then add to the ith row of D' multiplied by a_{ij} the first row multiplied by a_{ij}, the second multiplied by a_{2j}, . . ., the nth multiplied by a_{nj}, and show that $a_{ij}D' = 0$.

The case of $D = 0$ may be omitted if we assume the elements of D to be independent variables instead of numbers. Then the determinant will be a polynomial different from zero, and we prove that (1) is an identity; this means it is true for all numerical values of the variables a_{ij}, irrespective of whether D vanishes or not.

507. *Hint.* First consider the case where M lies in the upper left-hand corner. Multiplying with respect to rows of D the minor M' written in the form

$$\begin{vmatrix} A_{11} & \dots & A_{1m} & A_{1,\,m+1} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{m1} & \dots & A_{mm} & A_{m,\,m+1} & \dots & A_{mn} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{vmatrix},$$

show that $DM' = D^m A$ and $M' = D^{m-1}A$ (the case $D = 0$ may be omitted as indicated in the preceding problem, that is, by regarding D as a polynomial in n^2 unknowns a_{ij}). Then reduce the general arrangement of M to the one under consideration by permutations of rows and columns; for this purpose show that when two adjacent rows (or columns) are permuted, there occurs a similar permutation of rows (or columns) in the adjugate determinant D' and, besides, all elements of D' change sign.

508. *Hint.* Use the preceding problem.

509. *Hint.* Use problem 507.

510. *Hint.* Apply the equation of problem 507 for $m = n - 1$.

511. *Hint.* Apply the equation of problem 507 with $n - m$ substituted for m.

512. *Hint.* Use the value of the adjugate determinant D' to find the value of the determinant D and apply the equality of problem 510. Show that the problem has $n - 1$ solutions.

513. *Hint.* Express the first determinant as the square of the Vandermonde determinant made up of the numbers $0, x_1, x_2, \dots, x_n$.

514. $A_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$ $(i, j = 1, 2, \dots, n)$.

515. *Hint.* Consider the products $D\Delta$ and ΔD, where D is the given determinant and Δ is a determinant of the same order as D obtained by interchanging the ith and jth rows from a determinant having ones on the principal diagonal and zeros elsewhere.

516. *Hint.* Consider the products $D\Delta$ and ΔD, where D is the given determinant and in Δ the elements of the principal diagonal are equal to 1, the element in the ith row and in the jth column is equal to c, and all other elements are zero.

517. *Hint.* Set $\varphi_1 = \alpha_2 - \alpha_3$; $\varphi_2 = \alpha_3 - \alpha_1$, $\varphi_3 = \alpha_1 - \alpha_2$.

518. *Hint.* Apply the identity of problem 502.

520. The determinant is equal to twice the area of the triangle $M_1M_2M_3$ if the direction of the smallest rotation of the ray M_3M_1 to coincidence with M_3M_2 coincides with the direction of the smallest rotation from the positive direction of Ox to the positive direction of Oy, otherwise it is equal to twice the area of the triangle $M_1M_2M_3$ with the minus sign.

Solution. The indicated transformations of the coordinates are given by the formulas

$$x = x' \cos\alpha - y' \sin\alpha + x_0,\ y = x' \sin\alpha + y' \cos\alpha + y_0.$$

From this, multiplying with respect to rows, we obtain

$$\begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos\alpha & -\sin\alpha & x_0 \\ \sin\alpha & \cos\alpha & y_0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

But the second determinant in the left-hand member of the equation is equal to 1. This proves the invariability of the given determinant under the indicated transformations. Now carry the origin of coordinates to the point M_3 and rotate the axes so that the new axis of abscissas goes along M_3M_1. The new coordinates of the points M_1, M_2, M_3 will be $x_1' = M_3M_1$, $y_2' = \pm h$, where h is the altitude of the triangle $M_1M_2M_3$ dropped from the vertex M_1; the choice of the plus or minus sign is connected with the orientation of the triangle by the rule given above: $y_1' = x_3' = y_3' = 0$. For that reason the determinant takes the form

$$\begin{vmatrix} x_1' & 0 & 1 \\ x_2' & y_2' & 1 \\ 0 & 0 & 1 \end{vmatrix} = \pm M_3M_1 \cdot h = \pm 2S,$$

where S is the area of the triangle $M_1M_2M_3$.

521. The determinant is equal to the area of a parallelogram constructed on line segments joining the origin to the points M_1 and M_2, the area being taken with the plus sign if the directions of the shortest rotation from OM_1 to OM_2 and from Ox to Oy coincide, and with the minus sign if these directions are opposite. The determinant does not change under a rotation of the axes but may change under a translation of the origin. *Hint.* Apply the result of the preceding problem by taking the coordinate origin for the third point.

522. $R = \dfrac{abc}{4s}$. *Hint.* Take the centre of a circumscribed circle

for the coordinate origin and make use of the relations

$$R^2 - x_i x_j - y_i y_j = \frac{1}{2}\left[(x_i - x_j)^2 + (y_i - y_j)^2\right] \quad (i, j = 1, 2, 3)$$

and also the result of problem 520.

523. *Hint.* Show that the square of the determinant is equal to 1. To determine the sign, make use of the continuity of the determinant in the totality of all variables α_i, β_i, γ_i ($i = 1, 2, 3$), and show that it remains unchanged under a rotation of the figure $OABC$.

524. The determinant is equal to the volume of a parallelepiped constructed on line segments joining the coordinate origin O to the points M_1, M_2, M_3 or is equal to six volumes of the tetrahedron $OM_1M_2M_3$ taken with the plus sign if the orientations of the trihedrons $OM_1M_2M_3$ and $Oxyz$ are the same, and with the minus sign if these orientations are opposite (orientations are assumed to be the same if after coincidence—via rotation of the trihedron $Oxyz$—of the axis Ox with OM_1 and of the plane xOy with OM_1M_2 so that Oy and OM_2 lie to one side of Ox, the rays Oz and OM_3 lie to one side of the plane xOy; orientations are assumed to be opposite if the rays are on different sides). *Hint.* If α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, γ_3 are, respectively, the cosines of the new axes Ox', Oy', Oz' with the old axes, then the old and new coordinates are connected by the relations

$$x = \alpha_1 x' + \beta_1 y' + \gamma_1 z', \quad y = \alpha_2 x' + \beta_2 y' + \gamma_2 z',$$
$$z = \alpha_3 x' + \beta_3 y' + \gamma_3 z'.$$

Use this fact and the results of the preceding problem to prove the invariability of the determinant of the present problem. To determine the geometric meaning of the determinant, rotate the system of coordinates $Oxyz$ as indicated above for determining the like and opposite orientations of the trihedrons.

525. $V = abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}$. *Hint.* Compute V^2 making use of the result of the preceding problem.

526. *Hint.* On the rays OA, OB, OC, take the points M_1, M_2 and M_3 at a distance of 1 from the origin and apply the result of problem 524.

527. The determinant is equal to six volumes of a tetrahedron with vertices M_1, M_2, M_3, M_4 taken with the plus sign if the trihedron of rays from M_4 to each of the points M_1, M_2, M_3 has the same orientation as the trihedron $Oxyz$, and with the minus sign otherwise. *Hint.* Translate the origin to the point M_4 and apply the result of problem 524. An alternative way is to proceed as in the solution of problem 520, making use of problem 523. Then problem 524 will turn out to be a special case of the given problem (like what it was on the plane in problem 521).

528. $R = \frac{1}{24V} \times$

$$\times\sqrt{2l_{13}^2 l_{14}^2 l_{23}^2 l_{24}^2 + 2l_{12}^2 l_{14}^2 l_{32}^2 l_{34}^2 + 2l_{12}^2 l_{13}^2 l_{42}^2 l_{43}^2 - l_{12}^4 l_{34}^4 - l_{13}^4 l_{24}^4 - l_{14}^4 l_{23}^4},$$

where V is the volume of the tetrahedron and $l_{ij} = l_{ji}$ ($i, j = 1, 2, 3, 4$, $i \neq j$) is the length of the edge joining the vertices $(x_i\ y_i,\ z_i)$ and (x_j, y_j, z_j). In the case of a regular tetrahedron with edge a we obtain $R = \frac{a\sqrt{6}}{4}$.

Hint. Apply the result of the preceding problem and the relation

$$R^2 - x_i x_j - y_i y_j - z_i z_j = \frac{1}{2}[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2],$$

which is true on the assumption that the origin has been translated to the centre of a circumscribed sphere.

529. *Hint.* When proving statement (2), show that the vector $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ may be represented as $a_i = a_{i1}e_1 + a_{i2}e_2 + \ldots + a_{in}e_n$. Furthermore, show that the function $f(a_1, a_2, \ldots, a_n)$ changes sign when two vectors are interchanged. For example, when proving this with respect to the vectors a_1 and a_2, consider $f(a_1 + a_2, a_1 + a_2, a_3 \ldots a_n)$.

530. *Hint.* Prove that the function $f(a_1, a_2, \ldots, a_n) = |AB|$ of the rows of matrix A has the properties (α) and (β) and that $f(e_1, e_2, \ldots, e_n) = |B|$.

531. Set $f(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) = 1$ for arbitrary $i_1, i_2, \ldots, i_n = 1, 2, \ldots, n$ (identical or distinct). By virtue of (α), assuming $a_i = \sum_{j=1}^{n} a_{ij}e_j$, we get

$$f(a_1, a_2, \ldots, a_n) = \sum_{i_1, i_2, \ldots, i_n = 1}^{n} a_{1, i_1} a_{2, i_2} \cdots a_{n, i_n}.$$

This defines the function $f(a_1, a_2, \ldots, a_n)$. Clearly, it does not change under an interchange of vectors, that is, in the case of a field of characteristic 2 it changes sign. For that reason, (β') is fulfilled and (β) is obviously not fulfilled.

532. $(-1)^{C_n^2} i^{C_{n-1}^2} n^{\frac{n}{2}} = i^{\frac{(n-1)(3n-2)}{2}} n^{\frac{n}{2}}$. *Solution.* Multiplying the given determinant D by itself and noting that $\varepsilon^k = 1$ if and only if k is divisible by n, we obtain

$$D^2 = \begin{vmatrix} n & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & n \\ 0 & 0 & \ldots & n & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & n & \ldots & 0 & 0 \end{vmatrix} = (-1)^{C_{n-1}^2} n^n,$$

whence $D = \pm i^{C_{n-1}^2} n^{\frac{n}{2}}$; for the modulus of D we get $|D| = n^{\frac{n}{2}}$. It remains to determine the argument. Computing D as a Vandermonde determinant of the numbers $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1}$ and then setting $\varepsilon = \alpha^2$, where $\alpha = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, we obtain

$$D = \prod_{0 \leq j < k \leq n-1} (\varepsilon^k - \varepsilon^j) = \prod_{0 \leq j < k \leq n-1} (\alpha^{2k} - \alpha^{2j})$$

$$= \prod_{0 \leq j < k \leq n-1} \alpha^{k+j} (\alpha^{k-j} - \alpha^{-(k-j)})$$

$$= \prod_{0 \leq j < k \leq n-1} \alpha^{j+k} \prod_{0 \leq j < k \leq n-1} 2i \sin \frac{(k-j)\pi}{n}.$$

For the values of j and k considered above it is always true that $0 < k - j < n$ and, hence, $\sin \frac{(k-j)\pi}{n} > 0$. For that reason $|D| =$

$= \prod_{0 \leq j < k \leq n-1} 2 \sin \frac{(k-j)\pi}{n} = n^{\frac{n}{2}}$. $D = |D| \beta$ where $\beta = i^{C_n^2} \prod_{0 \leq j < k \leq n-1} \alpha^{j+k}$.

The exponent of α involves each integer p $(0 \leqslant p \leqslant n-1)$ exactly $n-1$ times: either as j for $k = p+1, p+2, \ldots, n-1$ or as k for $j = 0, 1, 2, \ldots, p-1$. Noting that $\alpha^{\frac{n}{2}} = i$ (when n is odd we have $\alpha^{\frac{n}{2}} = \pm i$, but the choice of sign plays no role because of the evenness of the number $n-1$; this is clear from the computations given below), we find

$$\beta = i^{\frac{n(n-1)}{2}} \alpha^{\frac{n(n-1)^2}{2}} = i^{\frac{n(n-1)}{2} + (n-1)^2} = i^{\frac{(n-1)(3n-2)}{2}}.$$

Setting $3n = 2n + n$ and using the expression for $|D|$ given above, we obtain the desired expression for D.

533. The determinant will be multiplied by $(2-k)\,2^{k-1}$. *Hint.* Reduce the case of any rows to the first rows. If the first k rows have been isolated, then the indicated transformation can be obtained by multiplying the given determinant on the left by a determinant of the same order in which all elements on the principal diagonal are equal to 1, the elements outside the principal diagonal in the first k rows and the first k columns are equal to -1, and all other elements are zero.

534. $D = D_0 + Sx$, where D_0 is the value of the determinant D for $x = 0$ and S is the sum of the cofactors of all elements of D_0.

535. *Hint.* Take advantage of the result of the preceding problem.

536. *Hint.* Use the preceding problem.

537. *Hint.* Put $x = -1$ in the determinant of problem 534.

539. *Hint.* Use the method of mathematical induction.

540. *Hint.* Split up all np rows of the determinant D into n sets of p rows each, putting in the first set all rows with numbers $1, n+1, 2n+1, \ldots, (p-1)n+1$, in the second set all rows with numbers $2, n+2, 2n+2, \ldots, (p-1)n+2, \ldots,$ in the nth set, all rows with the numbers $n, 2n, 3n, \ldots, pn$. Apply to these sets the generalized Laplace theorem of problem 466. Show that minors of order p of any of the indicated sets are equal to zero if at least two second indices of the elements b_{ij} are the same and that $D = f(a_{11}, \ldots, a_{nn}) B^n$, where $f(a_{11}, \ldots, a_{nn})$ does not depend on the elements b_{ij}. To determine $f(a_{11}, \ldots, a_{nn})$ put $b_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $b_{ij} = 0$ for $i, j = 1, 2, \ldots, n$, $i \neq j$.

542. *Solution.* If all $A_{ij} = 0$, then we can put

$$A_i = B_i = 0 \quad (i = 1, 2, \ldots, n).$$

For example, suppose the cofactors of the elements of the last column are not all zero (the reasoning is similar in the case of a different column). Since $D = 0$, the cofactors of the elements of two columns are proportional (see problem 509):

$$\frac{A_{1j}}{A_{1n}} = \frac{A_{2j}}{A_{2n}} = \ldots = \frac{A_{nj}}{A_{nn}} = \frac{B'_j}{C_j} \quad (j = 1, 2, \ldots, n-1),$$

where the fraction $\frac{B'_j}{C_j}$ will be regarded as irreducible; from this we have

$$A_{ij} = \frac{A_{in}B'_j}{C_j} \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n-1). \qquad (1)$$

But A_{ij} is a polynomial in $x_1, x_2, \ldots, x_s$ and the fraction $\frac{B'_j}{C_j}$ is irreducible. Therefore, A_{in} must be divisible by C_j $(j = 1, 2, \ldots, n-1)$ and, hence, by the least common multiple of all C_j $(j = 1, 2, \ldots, n-1)$. Denoting this least multiple by B_n, we obtain

$$A_{in} = A_i B_n \quad (i = 1, 2, \ldots, n), \qquad (2)$$

where all A_i are polynomials in $x_1, x_2, \ldots, x_s$. Set $B_j = \frac{B_n}{C_j} \cdot B'_j$ $(j = 1, 2, \ldots, n-1)$. All B_j are polynomials in $x_1, x_2, \ldots, x_s$ and from (1) we find

$$A_{ij} = A_i B_j \quad (i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n-1). \qquad (3)$$

The equations (2) and (3) prove the theorem. In particular, for the determinant Δ we can put $A_1 = B_1 = c$, $A_2 = B_2 = -b$, $A_3 = B_3 = a$.

543. *Solution.* Let D_{2n} be a skew-symmetric determinant of order $2n$. Carry out induction on n. For $n = 1$ the theorem holds because $D_2 = a_{12}^2$. Suppose the theorem holds true for the number n and then prove it for $n + 1$. Crossing out of the given determinant D_{2n+2} the last row and the last column, we get the skew-symmetric determinant D_{2n+1} equal to zero. Its elements may be regarded as polynomials in the elements lying above the principal diagonal with integer coefficients. By the theorem of the preceding problem, the cofactors of the elements of D_{2n+1} are of the form $A_{ij} = A_i B_j$ $(i, j = 1, 2, \ldots, 2n+1)$, where A_i and B_j are polynomials in the same unknowns. The minors M_{ij} and M_{ji} in D_{2n+1} are obtained one from the other by transposing and changing the signs of the elements, and since they are of even order $2n$, it follows that $A_{ij} = A_{ji}$. Or $A_i B_j = A_j B_i$.

$$\frac{B_i}{A_i} = \frac{B_j}{A_j} = \lambda; \quad B_i = \lambda \cdot A_i \quad (i = 1, 2, \ldots, 2n+1).$$

λ is a rational function of the same unknowns. Furthermore, $A_{ii} = A_i B_i = \lambda A_i^2$ and, by hypothesis, A_{ii} is a perfect square. Thus, $\lambda = \mu^2$ where μ is a rational function. And so $A_{ii} = (\mu A_i)^2$. On

the left we have a polynomial. But the square of an irreducible fraction cannot be a polynomial. Hence, $\mu A_i = c_i$ is a polynomial. Applying the expansion given in problem 541, we find

$$D_{2n+2} = -\sum_{i,\,j=1}^{2n+1} A_{ij}a_{i,\,2n+2}a_{2n+2,\,j} = \sum_{i,\,j=1}^{2n+1} A_iB_ja_{i,\,2n+2}a_{j,\,2n+2}$$

$$= \Big(\sum_{i=1}^{2n+1} A_ia_{i,\,2n+2}\Big)\Big(\sum_{j=1}^{2n+1} B_ja_{j,\,2n+2}\Big) = \lambda\Big(\sum_{i=1}^{2n+1} A_ia_{i,\,2n+2}\Big)^2$$

$$= \Big(\sum_{i=1}^{2n+1} C_ia_{i,\,2n+2}\Big)^2.$$

This proves the assertion of the problem. The proof does not yield a convenient technique for actually computing the polynomial whose square is equal to the given skew-symmetric determinant of even order. Such rules are given in problems 545 and 546.

544. *Hint.* Consider two terms in one of which the substitution of the indices has the cycle $(\alpha_1\alpha_2 \ldots \alpha_h)$ (h is odd and greater than unity) and in the other the cycle $(\alpha_h\alpha_{h-1} \ldots \alpha_2\alpha_1)$. Consider separately the case $h = 1$.

545. *Hint.* When proving (1) on the basis of the given pair N_1, N_2 of reduced Pfaffian products, restore the notation (1) of the substitution of the desired term, bearing in mind that if this term is written as

$$\pm a_{\alpha_1,\,\alpha_2}a_{\alpha_2,\,\alpha_3} \cdots a_{\alpha_h,\,\alpha_1}a_{\beta_1,\,\beta_2} \cdots a_{\mu_k,\,\mu_1},$$

then N_1' consists of elements occupying odd sites in the product, and N_2' consists of elements occupying even sites. For example, in N_1 we take the element with the first index $\alpha_1 = 1$. The second index α_2 yields the second element of the first cycle. In N_2 we take the element, one of the indices of which is α_2. If the other index is α_1, the cycle closes; if it is α_3, then this is the third term of the cycle, and so on. Show that all cycles in the resulting substitution are of even length. In the proof (2) note that $N_1 = N_1'$ and $N_2 = N_2'$. Determine the sign of the term as $(-1)^s$, where s is the number of cycles of the appropriate substitution. Derive assertion (3) from (1) and (2) and from the theorem of the preceding problem.

546. *Hint.* Show that each term of the Pfaffian p_n involves one and only one element of the nth column of D_n; in each term of p_n arrange the elements in increasing order of the second indices and show that if the element a_{in} is factored out of all terms that contain it, then the only thing left in the brackets will be p_{in} with the sign $(-1)^{n-1-i} = (-1)^{i-1}$.

547. $p_2 = a_{12}$; $p_4 = a_{23}a_{14} - a_{13}a_{24} + a_{12}a_{34}$;

$$p_6 = a_{34}a_{25}a_{16} - a_{24}a_{35}a_{16} + a_{23}a_{45}a_{16} - a_{34}a_{15}a_{26}$$
$$+ a_{14}a_{35}a_{26} - a_{13}a_{45}a_{26} + a_{24}a_{15}a_{36} - a_{14}a_{25}a_{36}$$
$$+ a_{12}a_{45}a_{36} - a_{23}a_{15}a_{46} + a_{13}a_{25}a_{46} - a_{12}a_{35}a_{46}$$
$$+ a_{23}a_{14}a_{56} - a_{13}a_{24}a_{56} + a_{12}a_{34}a_{56}.$$

548. $1 \cdot 3 \cdot 5 \ldots (n-1)$.

549. *Hint.* Expand, via the formula of problem 541, the determinant

$$D' = \begin{vmatrix} a_{11} & \dots & a_{1n} & x_1 \\ \dots & \dots & \dots & \dots \\ -a_{1n} & \dots & a_{nn} & x_n \\ -x_1 & \dots & -x_n & 0 \end{vmatrix}$$

obtained by bordering D and equate it to the square of the Pfaffian for D' by applying the formula of problem 546. In the resulting equation, put $x_i = x_j = 1$, $x_k = 0$ for $i \neq k \neq j$.

550. *Hint.* Using the fact that the product of two nonzero polynomials is nonzero, show that if $D = AB$ is the presumed factorization and some term of the polynomial A contains a_{11}, then no term B contains elements of the first row (or column). From this fact derive that no matter what the $i, j = 1, 2, \dots, n$, there is a term in A containing a_{1i} but no term in B contains a_{ij}.

551. *Hint.* When proving (2), determine $\bar{\Delta}_{n-k}$ by proceeding from the numbering $t_1, t_2, \dots, t_{\binom{n}{n-k}}$ of the combinations of the n numbers $1, 2, \dots, n$ taken $n-k$ at a time, which numbering is connected with the numbering $s_1, s_2, \dots, s_{\binom{n}{k}}$ that determines Δ_k so that t_i contains those $n-k$ numbers that do not appear in s_i. If σ_i is the sum of the numbers from the combination s_i, then take the factor $(-1)^{\sigma_i}$ out of the ith row and the ith column $\left(i = 1, 2, \dots, \binom{n}{n-k}\right)$ of the determinant $\bar{\Delta}_{n-k}$. When proving (4), show, using the equation of item (3) and the irreducibility of D established in the preceding problem and also the degree of D and Δ_k relative to the elements a_{ij}, that $\Delta_k = cD^{\binom{n-1}{k-1}}$, where c is independent of the elements a_{ij}. To determine c, show that both Δ_k and $D^{\binom{n-1}{k-1}}$ contain the term $(a_{11}a_{22} \dots a_{nn})^{\binom{n-1}{k-1}}$ with coefficient unity.

552. $P_n = Q_n = 1$. *Hint.* Show that $Q_n = P_n^2$.

553. *Hint.* Show that $d_{ij} = \sum_{k=1}^{n} p_{ki}p_{kj}\varphi(k)$, where p_{ij} are the same as in the preceding problem.

Chapter II. Systems of Linear Equations

554. $x_1 = x_2 = 1$, $x_3 = x_4 = -1$.
555. $x_1 = -2$, $x_2 = 0$, $x_3 = 1$, $x_4 = -1$.
556. $x_1 = 1$, $x_2 = x_3 = 2$, $x_4 = 0$.
557. $x_1 = 2$, $x_2 = -2$, $x_3 = 1$, $x_4 = -1$.
558. $x_1 = -0.4$, $x_2 = -1.2$, $x_3 = 3.4$, $x_4 = 1$.

559. $x=\frac{2}{3}$, $y=-1$, $z=\frac{3}{2}$, $t=0$.

560. $x=-3$, $y=0$, $z=-\frac{1}{2}$, $t=\frac{2}{3}$.

561. $x=2$, $y=-3$. $z=-\frac{3}{2}$, $t=\frac{1}{2}$.

562. The system does not have a solution. 563. The system does not have a solution.

564. Changing the numbering of the unknowns does not bring the system to an equivalent one, but when solving the system of equations it is permissible, provided that after the solution is complete we revert to the original numbering. *Hint.* Show that after the transformations of type (a), (b), (c), any equation of the new system can be expressed linearly in terms of an equation of the old system, and conversely.

567. $x_1=-1$, $x_2=3$, $x_3=-2$, $x_4=2$.

568. $x_1=2$, $x_2=1$, $x_3=-3$, $x_4=1$.

569. $x_1=-2$, $x_2=1$, $x_3=4$, $x_4=3$.

570. $x_1=0$, $x_2=2$, $x_3=\frac{1}{3}$, $x_4=-\frac{3}{2}$.

571. $x_1=\frac{1}{2}$, $x_2=-\frac{2}{3}$, $x_3=2$, $x_4=-3$.

572. $x_1=104\frac{6}{7}$, $x_2=7\frac{4}{7}$, $x_3=-10$, $x_4=1$.

573. $x_1=5$, $x_2=4$, $x_3=3$, $x_4=2$, $x_5=1$.

574. $x_1=3$, $x_2=-5$, $x_3=4$, $x_4=-2$, $x_5=1$.

575. $x_1=\frac{1}{2}$, $x_2=-2$, $x_3=3$, $x_4=\frac{2}{3}$, $x_5=-\frac{1}{5}$. *Hint.* For the new unknows take $2x_1$, $3x_4$, $5x_5$.

576. $x_1=5$, $x_2=4$, $x_3=-3$, $x_4=3$, $x_5=-2$.

577. $x_1=2$, $x_2=-\frac{3}{2}$, $x_3=4$, $x_4=3$, $x_5=\frac{5}{2}$.

578. The system is indeterminate, that is, it has an infinity of solutions; x_1 and x_2 can be expressed in terms of x_3 and x_4 thus: $x_1=6-26x_3+17x_4$, $x_2=-1+7x_3-5x_4$; note that x_3 and x_4 can assume any values.

579. The system is indeterminate. The general solution is $x_1=\frac{1}{10}(6-15x_2-x_4)$, $x_3=\frac{1}{5}(1+4x_4)$, where x_2 and x_4 assume arbitrary values.

580. The system is inconsistent, which means it does not have any solutions.

581. The system has no solutions.

584. If $a_1, a_2, \ldots, a_n$ are all elements of a field, then the polynomial $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$ is equal to zero as a function, but x^n has a coefficient of unity.

585. $f(x)=x^2-5x+3$. 586 $f(x)=2x^3-5x^2+7$.

587. For a given asymptotic direction, it is possible to draw a parabola (and only one) not above the nth degree through any $n+1$

distinct points in a plane, of which no two points lie on the straight line of the asymptotic direction.

588. $y=3x^3-5x^2+1$. 589. $x=y^4-3y^3-5y+5$.

590. $x=\frac{1}{4}(-a+b+c+d)$, $y=\frac{1}{4}(a-b+c+d)$,

$z=\frac{1}{4}(a+b-c+d)$, $t=\frac{1}{4}(a+b+c-d)$.

591. $x=\frac{1}{2}\left(-\frac{c-ad}{b-a}+\frac{c'-a'd}{b'-a'}+\frac{c''-a''d}{b''-a''}\right)$,

$y=\frac{1}{2}\left(\frac{c-ad}{b-a}-\frac{c'-a'd}{b'-a'}-\frac{c''-a''d}{b''-a''}\right)$,

$z=\frac{1}{2}\left(\frac{c-ad}{b-a}+\frac{c'-a'd}{b'-a'}-\frac{c''-a''d}{b''-a''}\right)$,

$t=d-\frac{1}{2}\left(\frac{c-ad}{b-a}+\frac{c'-a'd}{b'-a'}+\frac{c''-a''d}{b''-a''}\right)$.

Hint. To prove uniqueness, show that the determinant of the system is equal to $2(b-a)(b'-a')(b''-a'')\neq 0$. To find a solution, subtract from the first, second and third equations the fourth multiplied by a, a' and a'' respectively.

592. $x=\frac{1}{A}(ap-bq-cr-ds)$, $y=\frac{1}{A}(bp+aq-dr+cs)$,

$z=\frac{1}{A}(cp+dq+ar-bs)$, $t=\frac{1}{A}(dp-cq+br+as)$,

where $A=a^2+b^2+c^2+d^2$. *Hint.* Take advantage of problem 468.

593. $x_k=(-1)^kP_k$, where P_k is the sum of all possible products of k numbers $a_1, a_2, \ldots, a_n$. *Hint.* Take advantage of problem 346.

594. $x_k=\dfrac{\prod\limits_{i\neq k}(b-a_i)}{\prod\limits_{i\neq k}(a_k-a_i)}=\dfrac{f(b)}{(b-a_k)f'(a_k)}$, where $f(x)=(x-a_1)\times$ $\times(x-a_2)\ldots(x-a_n)$.

595.

$$x_k=(-1)^{n+k}\sum_{i=1}^{n}\frac{b_if_{ik}}{(a_i-a_1)\ldots(a_i-a_{i-1})(a_i-a_{i+1})\ldots(a_i-a_n)},$$

where f_{ik} is the sum of all possible products of $n-k$ numbers taken from the $n-1$ numbers $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$.

596.

$$x_k=\frac{1}{(a_k-a_1)\ldots(a_k-a_{k-1})(a_k-a_{k+1})\ldots(a_k-a_n)}\sum_{i=1}^{n}b_if_{ki},$$

where f_{kl} is the sum of all possible products of $n-l$ numbers taken from the $n-1$ numbers $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$.

597. $x_k = \dfrac{(-1)^k P_{n-k}}{n!}$, where P_l $(l = 1, 2, \ldots, n-1)$ is the sum of all possible products of l numbers taken from the n numbers $1, 2, \ldots, n$ and $P_0 = 1$.

598. $x_k = \dfrac{c_k}{a-b} - \dfrac{b \sum\limits_{i=1}^{n} c_i}{(a-b)[a+(n-1)b]}$.

599. $x_k = \prod\limits_{i \neq k} \dfrac{b-a_i}{a_k-a_i}$. *Hint.* Express the determinant of the system as a product of two determinants.

600. *Solution.* $\ln(1+x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 - \dfrac{1}{4}x^4 + \ldots$. Therefore $1 = (1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots)\left(1 - \dfrac{1}{2}x + \dfrac{1}{3}x^2 - \dfrac{1}{4}x^3 + \ldots\right)$, whence $h_1 = \dfrac{1}{2}$, $\dfrac{1}{2}h_1 - h_2 = \dfrac{1}{3}$, $\dfrac{1}{3}h_1 - \dfrac{1}{2}h_2 + h_3 = \dfrac{1}{4}$, $\dfrac{1}{4}h_1 - \dfrac{1}{3}h_2 + \dfrac{1}{2}h_3 - h_4 = \dfrac{1}{5}$, ... Using Cramer's rule, obtain the required expression for h_n from the first n equations.

602. *Hint.* Starting with the identity

$$1 = (1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots)\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots\right),$$

we obtain a system of equations for determining $b_1, b_2, b_3, \ldots$. To prove that $b_{2n-1} = 0$ when $n > 1$, note that $b_1 = -\dfrac{1}{2}$ and that function

$$\frac{x}{e^x - 1} - 1 + \frac{1}{2}x = \frac{\frac{1}{2}x\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right) - \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}$$

is even.

603. *Hint.* Using the equations $b_1 = -\dfrac{1}{2}$, $b_{2n-1} = 0$ for $n > 1$, obtained in the preceding problem, set $b_{2n} = c_n$ and in the identity

$$1 = \left(1 - \frac{1}{2}x + c_1 x^2 + c_2 x^4 + c_3 x^6 + \ldots\right)\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots\right)$$

equate separately the coefficients of even and odd powers of x.

604. *Hint.* To establish the required equality, put $x = 1, 2, 3, \ldots, k-1$ in the identity

$$(x+1)^n = x^n + C_n^{n-1}x^{n-1} + C_n^{n-2}x^{n-2} + \ldots + C_n^1 x + x^0$$

and add the resulting equalities. In the equality thus established replace n by $n-1, n-2, \ldots, 2, 1$ and find $s_{n-1}(k)$ from the resulting system of n linear equations in $s_0(k), s_1(k), \ldots, s_{n-1}(k)$.

605.

$$l_n = \begin{vmatrix} \frac{2}{3!} & 1 & 0 & \ldots & 0 \\ \frac{4}{5!} & \frac{1}{2!} & 1 & \ldots & 0 \\ \frac{6}{7!} & \frac{1}{4!} & \frac{1}{2!} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{2n}{(2n+1)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2!} \end{vmatrix}.$$

Hint. Obtain the identity

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) \times (1 + l_1 x^2 + l_2 x^4 + \ldots).$$

606.

$$f_n = \begin{vmatrix} \frac{2}{3!} & 1 & 0 & \ldots & 0 \\ \frac{4}{5!} & \frac{1}{3!} & 1 & \ldots & 0 \\ \frac{6}{7!} & \frac{1}{5!} & \frac{1}{3!} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{2n}{(2n+1)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \cdots & \frac{1}{3!} \end{vmatrix}.$$

607.

$$a_n = \frac{1}{n!} = \begin{vmatrix} \frac{1}{1!} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1 & 0 & \ldots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & \frac{1}{1!} \end{vmatrix}.$$

Hint. Use the identity

$$1 = (1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$$

to obtain equations in order to determine $a_1, a_2, \ldots, a_n$.

608. 2. 609. 3. 610. 3. 611. 2. 612. For $\lambda = 0$ the rank of the matrix is 2, for $\lambda \neq 0$ it is equal to 3. 613. For $\lambda = 3$ the rank is equal to 2, for $\lambda \neq 3$ it is 3. 619. 3. 620. 2. 621. 3. 622. 2. 629. *Hint.* Using the linear expression of all columns of matrix A in terms of the columns that pass through the minor d, show that if $d = 0$, the rows of the matrix A passing through d are linearly dependent.

630. If $0 \leqslant r \leqslant n-2$, then $\hat{r} = 0$. If $r = n-1$, then $\hat{r} = 1$. If $r = n$, then $\hat{r} = n$. *Hint.* Take advantage of problem 509 or 747.

631. *Solution.* We prove (1). For $r = 0$ all principal minors of the first and second orders are equal to zero. If $A = (a_{ij})_n$, then $a_{11} = a_{22} = \ldots = a_{nn} = 0$ and

$$\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = a_{ii}a_{jj} - a_{ij}^2 = -a_{ij}^2 = 0$$

for arbitrary i, $j = 1, 2, \ldots, n$; $i < j$. From this we have $a_{ij} = 0$, $i, j = 1, 2, \ldots, n$; $A = 0$; the rank of A is zero, which is what we set out to prove. For $r = n-1$ we have $M_{n-1} \neq 0$, $M_n = |A| = 0$ and the rank of A is equal to $n-1$.

Let $0 < r \leqslant n-2$. The principal minor $M_r \neq 0$. By interchanging the rows and columns of matrix A (this does not upset the symmetry of A or alter its rank), we can transfer the minor M_r to the upper left-hand corner of matrix A. When proving (1), it suffices to show that all minors of order $r+1$ that border M_r are zero.

Let M_{ij} be a minor obtained from M_r by bordering it with the ith row and the jth column ($i, j > r$). By hypothesis, $M_{ij} = 0$ when $i = j$. Let $i \neq j$ and let D be the determinant obtained from M_r by bordering it with the ith and jth rows and the ith and jth columns. By hypothesis, $D = 0$. Let C be the matrix of the determinant D. Suppose $M_{ij} \neq 0$. Then the rank of C is equal to $r+1$ and the rows of matrix C with labels $1, 2, \ldots, r, i$ are linearly independent. By the symmetry of C, the columns with the same labels are also linearly independent. By problem 629, the minor M_{ii} located at the intersection of these rows and columns is nonzero, but this runs counter to the hypothesis. Assertion (2) follows from (1) or directly from problem 629.

632. *Hint.* Use the preceding problem.

633. *Hint.* Use the solution of problem 631.

634. *Hint.* Take advantage of the preceding problem.

636. (1, 4, −7, 7). 637. $x = (0, 1, 2, -2)$. 638. $x = (1, 2, 3, 4)$. 639. Linearly independent. 640. Linearly dependent. 641. Linearly independent. 642. Linearly dependent. 643. Linearly dependent. 644. Linearly independent.

651. *Hint.* Assuming that $\sum_{i=1}^{s} \lambda_i a_i = 0$, where not all λ_i are zero, and taking, from among the λ_i, the largest (in absolute value) coefficient λ_j, show that the jth coordinate of the linear combination thus chosen is different from zero.

656. *Hint.* On the assumption that two vectors a_i, a_j ($i > j$) can be expressed linearly in terms of the preceding ones, find an expression for the vector b from the expression of a_j and substitute that expression into the expression of a_i.

657. *Hint.* In the set of vectors $a_1, a_2, \ldots, a_r$ adjoin b_1 on the left and delete the vectors that are linearly expressible in terms of preceding vectors; then adjoin b_2 on the left and again delete the vectors that can be expressed linearly in terms of the preceding vectors, and so forth.

Take advantage of the preceding problem.

658. *Hint.* Make use of problems 653 and 657.

659. *Hint.* Assuming the given subsystem to be ordered, adjoin to the right of the system all vectors of the system and delete all vectors that can be linearly expressed in terms of the preceding ones.

662. It is impossible to choose such numbers.

664. *Hint.* When proving (3), make use of problem 663 and also problem 658 (item (c)).

665. $\lambda = 15$. 666. λ is any number. 667. λ is any number.

668. λ is not equal to 12. 669. There is no such value of λ.

670. The vectors a_1, a_2, a_3 in problem 665 are coplanar (lie in one plane) but are not collinear (they do not lie on one line). When $\lambda = 15$, vector b lies in the same plane and is expressed in terms of a_1, a_2, a_3, but when $\lambda \neq 15$ it does not lie in that plane and cannot be expressed in terms of those vectors. In problem 666 the vectors a_1, a_2, a_3 are not coplanar and any vector of three-dimensional space can be expressed linearly in terms of them. In problem 667 the vectors a_1, a_2 are not collinear and lie in the plane $4x_1 - 3x_2 = 0$.

For any value of λ, the vector b lies in the same plane and can be expressed linearly in terms of a_1, a_2.

In problem 668, the vectors a_1, a_2 are not collinear and the vector b is not coplanar with a_1 and a_2. When $\lambda = 12$, the vector a_3 is coplanar with a_1 and a_2 and the vector b cannot be expressed in terms of a_1, a_2, a_3. When $\lambda \neq 12$, the vectors a_1, a_2, a_3 are not coplanar and b can be expressed in terms of them.

In problem 669, the vectors a_1, a_2, a_3 lie in the plane $3x_2 - x_3 = 0$. When λ varies between $-\infty$ and $+\infty$, the terminus of vector b describes the straight line $x_1 = 2, x_2 = 5$ that is parallel to that plane. The vector b does not lie in the indicated plane for any value of λ and it cannot be expressed in terms of a_1, a_2, a_3.

672. There are four such systems: (1) a_1, a_3; (2) a_1, a_4; (3) a_2, a_3; (4) a_2, a_4.

673. (1) a_1, a_2; (2) a_2, a_3.

674. Any two vectors form a basis.

675. (1) a_1, a_4; (2) a_2, a_4; (3) a_3, a_4.

676. Any three vectors, except a_1, a_2, a_5 and a_2, a_4, a_5, form a basis.

677. The only basis occurs if and only if either the entire system coincides with the basis or all vectors of the system that do not appear in the basis are zero.

678. Two bases.

679. For example, the basis is formed by the vectors a_1, a_2, a_4; $a_3 = a_1 - a_2$.

680. One of the bases is formed by the vectors a_1, a_2, a_3; $a_4 = 2a_1 - 3a_2 + 4a_3$, $a_5 = a_1 + 5a_2 - 5a_3$.

681. One of the bases is formed by the vectors a_1, a_2, a_5; $a_3 = a_1 - a_2 + a_5$, $a_4 = 3a_1 + 4a_2 - 2a_5$.

682. *Hint.* When proving (1), put into $\sum_{j=1}^{n} \alpha_j x_j = 0$ the expressions $x_i = \sum_{j=1}^{r} \lambda_{ij} x_j$ $(i = r+1, \ r+2, \ldots, n)$ and show that $\alpha_j = -\sum_{i=r+1}^{n} \alpha_i \lambda_{i,j}$, $j = 1, 2, \ldots, r$.

Using this fact, show that if $a_i = (-\lambda_{i,1}, -\lambda_{i,2} \ldots, \lambda_{i,r}, 0, \ldots \ldots, 1, \ldots, 0)$, $i = r+1, r+2, \ldots, n$, where the coordinate equal to unity occupies the $(r+i)$th position and $a = (\alpha_1, \alpha_2, \ldots \ldots, \alpha_n)$, then $a = \sum_{i=r+1}^{n} \alpha_i a_i$.

When proving (2) to (4), make use of problem 664.

683. $2f_1 - 3f_2 - f_3 = 0$, $f_1 - 3f_2 + f_4 = 0$.

684. $f_1 - 2f_2 - f_3 = 0$, $f_1 - 3f_2 + f_4 = 0$, $3f_1 - 8f_2 - f_5 = 0$.

685. The forms are linearly independent. No basic system of linear relations exists.

686. $2f_1 - f_2 - f_3 = 0$, $2f_1 - 2f_2 + f_4 - f_5 = 0$.

687. $2f_1 - f_2 = 0$, $f_1 - f_2 - f_4 + f_5 = 0$.

688. *Hint.* Make use of problem 661 or 657.

689. For example, the general solution is $x_1 = \frac{x_3 - 9x_4 - 2}{11}$, $x_2 = \frac{-5x_3 + x_4 + 10}{11}$. A particular solution is $x_1 = -1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$.

690. For example, the general solution is $x_3 = 22x_1 - 33x_2 - 11$, $x_4 = -16x_1 + 24x_2 + 8$. A particular solution is $x_1 = 2$, $x_2 = 1$, $x_3 = x_4 = 0$.

691. General solution: $x_3 = 1 - 3x_1 - 4x_2$, $x_4 = 1$. A particular solution: $x_1 = -1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$.

692. The system is inconsistent.

693. The system has a unique solution: $x_1 = 3$, $x_2 = 2$, $x_3 = 1$.

694. General solution: $x_3 = 6 - 15x_1 + 10x_2$, $x_4 = -7 + 18x_1 - 12x_2$. A particular solution: $x_1 = x_2 = x_3 = 1$, $x_4 = -1$.

695. General solution: $x_3 = \frac{34x_1 - 17x_2 - 29}{5}$, $x_4 = \frac{16x_1 - 8x_2 - 16}{5}$. A particular solution: $x_1 = 2$, $x_2 = 1$, $x_3 = \frac{22}{5}$, $x_4 = \frac{8}{5}$.

696. General solution: $x_3 = 2 - \frac{27}{13} x_1 + \frac{9}{13} x_2$, $x_4 = -1 + \frac{3}{13} x_1 - \frac{1}{13} x_2$. A particular solution: $x_1 = x_2 = 1$, $x_3 = \frac{8}{13}$, $x_4 = -\frac{11}{13}$.

697. General solution: $x_1 = \frac{-6 + 8x_4}{7}$, $x_2 = \frac{1 - 13x_4}{7}$, $x_3 =$

$= \frac{15-6x_4}{7}$. A particular solution: $x_1=-2, x_2=2, x_3=3, x_4=-1$.

698. The system is inconsistent.

699. General solution: $x_3=-1-8x_1+4x_2$, $x_4=0$, $x_5=1+2x_1-x_2$. A particular solution: $x_1=1$, $x_2=2$, $x_3=-1$, $x_4=0$, $x_5=1$.

700. General solution: $x_3=13$, $x_4=19-3x_1-2x_2$, $x_5=-34$. A particular solution: $x_1=1$, $x_2=8$, $x_3=13$, $x_4=0$, $x_5=-34$.

701. General solution: $x_3=-\frac{9}{2}-x_1-2x_2$, $x_4=-\frac{25}{2}-2x_1-4x_2$, $x_5=-\frac{15}{2}-2x_1-4x_2$. A particular solution: $x_1=1$, $x_2=-3$, $x_3=\frac{1}{2}$, $x_4=-\frac{5}{2}$, $x_5=\frac{5}{2}$.

702. General solution: $x_3=\frac{4}{3}x_1+\frac{2}{3}x_2$, $x_4=-\frac{14}{3}x_1-\frac{7}{3}x_2-1$, $x_5=\frac{4}{3}x_1+\frac{2}{3}x_2+2$. A particular solution: $x_1=x_2=1$, $x_3=2$, $x_4=-8$, $x_5=4$.

703. The system has a unique solution: $x_1=3$, $x_2=0$, $x_3=-5$, $x_4=11$.

704. The system is inconsistent.

705. *Hint.* When proving the minimality of the number $n-r$ of free unknowns, take advantage of the relationship between the solutions of a nonhomogeneous and the corresponding homogeneous system of equations and the fact that the number of unknowns of the homogeneous system that may assume arbitrary (unrelated) values is equal to the maximal number of linearly independent solutions and, hence, is independent of the choice of those unknowns.

706. $x_1=\frac{1}{2}x_4+\frac{31}{6}$, $x_2=\frac{2}{3}$, $x_3=-\frac{1}{2}x_4-\frac{7}{6}$.

707. $x_1=x_4-\frac{53}{18}x_5+\frac{20}{9}$, $x_2=-\frac{5}{2}x_4+\frac{5}{6}x_5-\frac{5}{3}$, $x_3=\frac{2}{9}x_5-\frac{1}{9}$.

708. The systems are inconsistent.

709. The system has a unique solution: $x_1=1$, $x_2=2$, $x_3=-4$, $x_4=-3$.

710. $x_1=-\frac{12}{205}$, $x_2=\frac{176}{123}+\frac{4}{3}x_3$, $x_4=\frac{97}{205}$. Here, x_3 is a free unknown.

711. $x_1=-\frac{1}{2}-\frac{7}{12}x_2-\frac{5}{4}x_3-\frac{7}{8}x_5$, $x_4=1-\frac{1}{2}x_5$, where x_2, x_3, x_5 are free unknowns.

712. When $\lambda \neq 0$ the system is inconsistent. When $\lambda=0$ it is consistent and the general solution is of the form

$$x_1=\frac{-5x_3-13x_4-3}{2}, \quad x_2=\frac{-7x_3-19x_4-7}{2}.$$

713. When $\lambda=0$ the system is inconsistent. When $\lambda \neq 0$ it is consistent and the general solution is of the form $x_1=\frac{4-\lambda}{5\lambda}-$ $-\frac{3}{5}x_3$, $x_2=\frac{9\lambda-16}{5\lambda}-\frac{8}{5}x_3$, $x_4=\frac{1}{\lambda}$.

714. When $\lambda = 1$ the system is inconsistent. When $\lambda \neq 1$ it is consistent and the general solution is of the form

$$x_1=\frac{43-8\lambda}{8-8\lambda}-\frac{9}{8}x_3,\ x_2=\frac{5}{4-4\lambda}+\frac{1}{4}x_3,\ x_4=\frac{5}{\lambda-1}.$$

715. The system is consistent for any value of λ. When $\lambda = 8$ the general solution is of the form $x_2 = 4 + 2x_1 - 2x_4$, $x_3 = 3 - 2x_4$, where x_1, x_4 are free unknowns. When $\lambda \neq 8$ the general solution is of the form $x_1 = 0$, $x_2 = 4 - 2x_4$, $x_3 = 3 - 2x_4$, where x_4 is a free unknown.

716. The system is consistent for arbitrary values of λ. When $\lambda = 8$ the general solution is of the form $x_3=-1$, $x_4 = 2 - x_1 - \frac{3}{2}x_2$, where x_1, x_2 are free unknowns. When $\lambda \neq 8$, the general solution is of the form $x_2=\frac{4}{3}-\frac{2}{3}x_1$, $x_3 = -1$, $x_4 = 0$, where x_1 is a free unknown.

717. When $(\lambda - 1)(\lambda + 2) \neq 0$, the system has a unique solution: $x_1 = x_2 = x_3 = \frac{1}{\lambda+2}$. When $\lambda = 1$ the general solution is of the form $x_1 = 1 - x_2 - x_3$, where x_2 and x_3 are free unknowns. When $\lambda = -2$ the system is inconsistent.

718. When $(\lambda - 1)(\lambda + 3) \neq 0$ the system has a unique solution: $x_1 = x_2 = x_3 = x_4 = \frac{1}{\lambda+3}$. When $\lambda = 1$ the general solution is of the form $x_1 = 1 - x_2 - x_3 - x_4$, where x_2, x_3, x_4 are free unknowns. When $\lambda = -3$ the system is inconsistent.

719. When $\lambda(\lambda+3) \neq 0$ the system has a unique solution: $x_1=\frac{2-\lambda^2}{\lambda(\lambda+3)}$, $x_2=\frac{2\lambda-1}{\lambda(\lambda+3)}$, $x_3=\frac{\lambda^3+2\lambda^2-\lambda-1}{\lambda(\lambda+3)}$. When $\lambda=0$ and $\lambda=-3$ the system is inconsistent.

720. When $\lambda(\lambda + 3) \neq 0$ the system has a unique solution: $x_1 = 2 - \lambda^2$, $x_2 = 2\lambda - 1$, $x_3 = \lambda^3 + 2\lambda^2 - \lambda - 1$. When $\lambda = 0$ the general solution is of the form $x_1 = -x_2 - x_3$, where x_2, x_3 are free unknowns. When $\lambda = -3$ the general solution is of the form $x_1 = x_2 = x_3$, where x_3 is a free unknown.

721. If a, b, c are pairwise distinct, the system has a unique solution:

$$x=\frac{(b-d)(c-d)}{(b-a)(c-a)},\quad y=\frac{(d-a)(d-c)}{(b-a)(b-c)},\quad z=\frac{(d-a)(d-b)}{(c-a)(c-b)}.$$

If there are only two distinct numbers among a, b, c, d, and $a \neq b$ or $a \neq c$ or $b \neq c$, then the solution depends on one parameter. For example: in the case of $d = a \neq b = c$ the general solution

is of the form $x = \frac{b-d}{b-a} = 1$, $y = \frac{a-c}{b-a} z$, where z is a free unknown that plays the part of the above-mentioned parameter which determines the solution.

If $a = b = c = d$, then the solution depends on two parameters and the general solution is, say, of the form $x = 1 - y - z$, where y, z are free unknowns. If there are two distinct numbers from among a, b, c and if d is not equal to any one of them or if $a = b = c \neq d$, then the system is inconsistent.

722. When $D = abc - a - b - c + 2 \neq 0$ the system has a unique solution:

$$x = \frac{(b-1)(c-1)}{D}; \quad y = \frac{(a-1)(c-1)}{D}; \quad z = \frac{(a-1)(b-1)}{D}.$$

In this case, any two of the unknowns can have zero values at the same time, while the third unknown and the corresponding parameter are equal to unity. For example, $x = y = 0$, $z = c = 1$. If $D = 0$, and if one and only one of the numbers a, b, c is different from unity, then the solution depends on a single parameter: for example, when $a \neq b = c = 1$ the general solution is of the form $x = 0$, $y = 1 - z$. In this case, one or two of the unknowns must be zero.

If $a = b = c = 1$ the general solution is of the form $x = 1 - y - z$ and one or two of the unknowns may be zero. If $D = 0$ and none of the numbers a, b, c is equal to unity, then the system is inconsistent. The case of $D = 0$ with one and only one of the numbers a, b, c equal to unity is impossible.

723. If $D = abc - a - b - c + 2 \neq 0$, the system has a unique solution:

$$x = \frac{abc - 2bc + b + c - a}{D}, \quad y = \frac{abc - 2ac + a + c - b}{D},$$

$$z = \frac{abc - 2ab + a + b - c}{D}.$$

If $D = 0$ and only one of the numbers a, b, c is different from unity, then the solution depends on a single parameter, for example, when $a \neq b = c = 1$ the general solution is of the form $x = 1$, $y = -z$, where z is a free unknown. If $a = b = c = 1$, then the solution depends on two parameters and the general solution is of the form $x = 1 - y - z$, where y, z are free unknowns. If $D = 0$, and all numbers a, b, c are different from unity, the system is inconsistent. The case of $D = 0$ with only one of the numbers a, b, c being equal to unity is impossible. *Hint.* In order to prove the inconsistency of the system when $D = 0$ under the condition that all numbers a, b, c are different from unity, show that the following identities hold true:

$$D - D_x = 2(b-1)(c-1), \quad D - D_y = 2(a-1)(c-1),$$
$$D - D_z = 2(a-1)(b-1),$$

where D_x, D_y, D_z are the appropriate numerators in the above expressions for x, y, z.

724. For example, the general solution is $x_1 = 8x_3 - 7x_4$, $x_2 = -6x_3 + 5x_4$. The fundamental set of solutions is

x_1	x_2	x_3	x_4
8	−6	1	0
−7	5	0	1

725. General solution: $x_2 = -\frac{5}{2}x_1 + 5x_3$, $x_4 = \frac{7}{2}x_1 - 7x_3$. The fundamental solution set is

x_1	x_2	x_3	x_4
1	0	$-\frac{5}{2}$	$+\frac{7}{2}$
0	1	+5	−7

726. General solution: $x_4 = -\frac{9x_1 + 6x_2 + 8x_3}{4}$, $x_5 = -\frac{3x_1 + 2x_2 + 4x_3}{4}$. The fundamental solution set is

x_1	x_2	x_3	x_4	x_5
1	0	0	$-\frac{9}{4}$	$\frac{3}{4}$
0	1	0	$-\frac{3}{2}$	$\frac{1}{2}$
0	0	1	−2	1

727. The system only has a zero solution. There is no fundamental solution set.

728. General solution: $x_4 = \frac{-9x_1 + 3x_2 - 10x_3}{11}$, $x_5 = \frac{-3x_1 + x_2 + 4x_3}{11}$. The fundamental solution set is

x_1	x_2	x_3	x_4	x_5
1	0	0	$-\frac{9}{11}$	$-\frac{3}{11}$
0	1	0	$\frac{3}{11}$	$\frac{1}{11}$
0	0	1	$-\frac{10}{11}$	$\frac{4}{11}$

729. The system has only a zero solution.

730. General solution: $x_1 = x_4 - x_5$, $x_2 = x_4 - x_6$, $x_3 = x_4$. The fundamental solution set is

x_1	x_2	x_3	x_4	x_5	x_6
1	1	1	1	0	0
−1	0	0	0	1	0
0	−1	0	0	0	1

731. General solution: $x_1 = 0$, $x_2 = \frac{x_3 - 2x_5}{3}$, $x_4 = 0$, where x_3, x_5 are free unknowns. The fundamental solution set is

x_1	x_2	x_3	x_4	x_5
0	$\frac{1}{3}$	1	0	0
0	$-\frac{2}{3}$	0	0	1

732. General solution: $x_1 = -3x_3 - 5x_5$, $x_2 = 2x_3 + 3x_5$, $x_4 = 0$, where x_3, x_5 are free unknowns. The fundamental solution set is

x_1	x_2	x_3	x_4	x_5
−3	2	1	0	0
−5	3	0	0	1

735. $x_1 = 13c$, $x_2 = 2c$, $x_3 = 7c$.

736. $x_1 = 4c_1$, $x_2 = 8c_2$, $x_3 = -3c_1 + 3c_2$, $x_4 = c_1 - c_2$.

737. $x_1 = 2c_1$, $x_2 = c_2$, $x_3 = 3c_2$, $x_4 = -3c_1 - c_2 - 4c_3$, $x_5 = -c_3$.

738. $x_1 = 14c_1$, $x_2 = 42c_2$, $x_3 = 42c_3$, $x_4 = -3c_1 + 3c_2 - 9c_3$, $x_5 = -8c_1 + 8c_2 - 10c_3$.

739. $x_1 = c_1 - 7c_2$, $x_2 = 2c_1 + 6c_2$, $x_3 = c_1 + 3c_2$, $x_4 = -4c_1$, $x_5 = -5c_2$.

740. $x_1 = 11c_1$, $x_2 = 33c_2$, $x_3 = -24c_1 - 57c_2$, $x_4 = 5c_1 + 5c_2$, $x_5 = -c_1 - c_2$.

741. The rows of matrix A do not form a fundamental solution set, the rows of matrix B do.

742. The fourth row together with any two of the first three rows forms a fundamental set, while the other systems of rows do not form a fundamental set.

743. *Hint.* In the first part of the problem apply the result of problem 734. In the second part of the problem show that if the values of the free unknowns yield, in a certain solution set, linearly dependent rows, then the whole solution set is linearly dependent.

748. *Hint.* Adjoin to the top of the matrix of the system any one of the rows and then expand the determinant of the resulting matrix in terms of the first row. Make use of problem 746.

749. A particular solution: $x_1 = -2$, $x_2 = -6$, $x_3 = 7$. The general solution is $x_1 = -2c$, $x_2 = -6c$, $x_3 = 7c$.

750. A particular solution: $x_1 = 3$, $x_2 = -2$, $x_3 = 0$. The general solution is $x_1 = 3c$, $x_2 = -2c$, $x_3 = 0$.

751. A particular solution: $x_1 = -6$, $x_2 = 11$, $x_3 = -9$, $x_4 = 4$. The general solution is $x_1 = 6c$, $x_2 = -11c$, $x_3 = 9c$, $x_4 = -4c$.

752. A particular solution: $x_1 = 3$, $x_2 = 0$, $x_3 = 4$, $x_4 = 0$. The general solution is $x_1 = 3c$, $x_2 = 0$, $x_3 = 4c$, $x_4 = 0$.

754. (a) $\sum_{j=1}^{n} a_{ij}x_j = 2b_i \quad (i = 1, 2, \ldots, s)$;

(b) $\sum_{j=1}^{n} a_{ij}x_j = \lambda b_i \quad (i = 1, 2, \ldots, s)$.

755. In both cases the necessary and sufficient condition is the homogeneity of the given system.

756. Provided that the sum of the coefficients of the given linear combination is equal to unity.

757. The first unknown in any solution assumes the value zero. If the coefficients of all unknowns, except the first and, for example, the second, are equal to zero, then the second unknown takes on a definite value that can be found from an equation involving a nonzero coefficient of the second unknown if all terms involving other unknowns are dropped; in that case, all unknowns from the third onwards may assume arbitrary values. But if at least three unknowns (say x_1, x_2, and x_3) occur with nonzero coefficients, then all unknowns, except the first, may assume arbitrary values, and their values in each solution are connected by a single relation obtained from any equation of the system that contains a nonzero coefficient of the second unknown, if the term involving the first unknown is dropped.

Under the conditions of the problem, it is impossible that all coefficients of the first unknown or of all unknowns, beginning with the second, can be equal to zero.

758. A necessary and sufficient condition for this is that the rank of the matrix of coefficients of the unknowns be reduced by unity when the kth column is deleted, in other words that the kth column not be a linear combination of the other columns of the matrix.

759. The rank of the augmented matrix (made up of the coefficients of the unknowns and the constant terms) must be reduced by unity when the kth column is deleted.

760. $e = ad - bc = 0$.

761. One condition stating that the determinant D of order r is not equal to zero, and $(s - r)(n - r + 1)$ conditions stating that the determinants of order $r + 1$ bordering D are equal to zero.

The latter conditions are independent since each contains an element that does not appear in the other conditions; it stands at the intersection of the bordering row and column and has a factor $D \neq 0$.

762. Either at least two of the numbers a, b, c, d, e are equal to -1 or none is equal to -1, but then

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{d}{d+1}+\frac{e}{e+1}=1.$$

763. $\lambda = af + bg + ch = 0$. *Hint.* Add all equations after multiplying them, respectively, by λx, λy, λz, λt. After obtaining the condition $\lambda = 0$, the determinant of the system may be computed as a skew-symmetric determinant following problem 547.

764. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

765. $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

766. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$; this condition is sufficient if, in the case of three parallel lines, their common point is taken to be the ideal point (point at infinity) of the given direction. If ideal points are excluded, then the necessary and sufficient condition is the equality of the ranks of two matrices: $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$ and $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$.

767. The rank of the matrix $\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \cdot & \cdot & \cdot \\ x_n & y_n & 1 \end{pmatrix}$ must be less than three.

768. Assuming ideal points, the rank of the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \cdot & \cdot & \cdot \\ a_n & b_n & c_n \end{pmatrix}$

must be less than three. If we do not assume ideal points, the rank of the reduced matrix must coincide with the rank of the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \cdots & \cdots \\ a_n & b_n \end{pmatrix}$.

769. $\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix}=0.$

770. $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0.$

771. $x^2+y^2-4x-1=0$. The centre is at the point (2, 0). The radius is equal to $\sqrt{5}$.

772. *Hint.* Use the answer to problem 770.

773. $\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix}=0.$

774. The hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$.

775. $2x^2+7y^2+y-8=0$. This is an ellipse with centre at the point $\left(0, -\frac{1}{14}\right)$ and with semi-axes of length $\frac{15}{28}\sqrt{14}$ and $\frac{15}{14}$; the major axis is parallel to the axis of abscissas and the minor axis lies on the axis of ordinates.

776. $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$

777. $\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 3 & -1 & -1 & 1 \end{vmatrix}=0,$

or $4x+y+3z-8=0$.

778. If one assumes ideal points (points at infinity), then

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=0.$$

But if one does not assume ideal points, the rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

must not change when the last column is deleted.

779. The rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & d_n \end{pmatrix}$$

is equal to two and does not change when the last column is deleted.

780. $$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

781. $x^2 + y^2 + z^2 - x - 2 = 0$. The centre lies at the point $\left(\frac{1}{2}, 0, 0\right)$. The radius is equal to 3/2.

782. A system of three linear equations in two unknowns; in this system, the augmented matrix and the three matrices of the coefficients of the unknowns for any pair of equations all have a rank of 2.

783. A system of three linear equations in two unknowns; in this system, the ranks of the matrices of the coefficients of the unknowns in any pair of equations are equal to two, and the rank of the augmented matrix is equal to three.

784. A system of three equations in three unknowns; in this system, the ranks of all matrices made up of the coefficients of the unknowns of any two equations and also of all three equations are equal to two, and the rank of the augmented matrix is equal to three.

785. A system of four linear equations in three unknowns; in this system, the ranks of the matrices made up of the coefficients of the unknowns of any three equations are equal to three, and the rank of the augmented matrix is equal to 4.

786. Four planes pass through one point, but no three of them pass through one straight line.

787. If one does not consider ideal straight lines and planes (lines and planes at infinity), then equations of the form $0x + 0y = a$ and $0x + 0y + 0z = a$, where a is nonzero, are geometrically meaningless, but when $a = 0$ they are satisfied by the coordinates of any point in the plane or in space. By eliminating equations of that type and designating the rank of the matrix made up of the coefficients

of the unknowns by r and the rank of the augmented matrix by r_1, we have the following.

For systems in two unknowns:

1. $r = 2$, $r_1 = 3$. The system has no solutions. The straight lines do not pass through one point, and at least two straight lines are distinct and intersect.

2. $r = r_1 = 2$. The system has a unique solution. The straight lines pass through one point, and at least two are distinct.

3. $r = 1$, $r_1 = 2$. The system has no solutions. The straight lines are parallel or are coincident, and at least two straight lines are distinct.

4. $r = r_1 = 1$. The solution depends on one parameter. All straight lines are coincident.

For systems in three unknowns:

1. $r = 3$, $r_1 = 4$. The system has no solutions. The planes do not pass through one point, and at least three of them are distinct and pass through one point.

2. $r = r_1 = 3$. The system has a unique solution. The planes pass through one point, and at least three of them do not pass through one straight line.

3. $r = 2$, $r_1 = 3$. The system has no solutions. The planes do not pass through one point, and at least three planes are distinct and any three distinct planes either have no point in common or pass through one straight line.

4. $r = r_1 = 2$. The solution depends on one parameter. All planes pass through one straight line, and at least two of them are distinct.

5. $r = 1$, $r_1 = 2$. The system has no solutions. The planes are parallel or coincide, and at least two of them are distinct.

6. $r = r_1 = 1$. The solution depends on two parameters. All planes are coincident.

Chapter III. Matrices and Quadratic Forms

788. $\begin{pmatrix} 5 & 2 \\ 7 & 0 \end{pmatrix}$.

789. $\begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$.

790. $\begin{pmatrix} 1 & 5 & -5 \\ 3 & 10 & 0 \\ 2 & 9 & -7 \end{pmatrix}$.

791. $\begin{pmatrix} 11 & -22 & 29 \\ 9 & -27 & 32 \\ 13 & -17 & 26 \end{pmatrix}$.

792. $\begin{pmatrix} 10 & 17 & 19 & 23 \\ 17 & 23 & 27 & 35 \\ 16 & 12 & 9 & 20 \\ 7 & 1 & 3 & 10 \end{pmatrix}$.

793. $\begin{pmatrix} 8 & 6 & 4 & 2 \\ 5 & 0 & -5 & -10 \\ 7 & 7 & 7 & 7 \\ 10 & 9 & 8 & 7 \end{pmatrix}$.

794. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

795. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

796. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

797. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

798. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

799. $\begin{pmatrix} 13 & -14 \\ 21 & -22 \end{pmatrix}$.

800. $\begin{pmatrix} 304 & -61 \\ 305 & -62 \end{pmatrix}$.

801. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for n even, $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ for n odd.

802. $\begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$.

803. $\begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$.

804. $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

805. $\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

806. $\begin{pmatrix} 1 & 3 & 6 & \dots & \frac{n(n+1)}{2} \\ 0 & 1 & 3 & \dots & \frac{(n-1)n}{2} \\ 0 & 0 & 1 & \dots & \frac{(n-2)(n-1)}{2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$, where n is the order of the given matrix.

807. $\begin{pmatrix} 1 & C_{n-1}^1 & C_{n-1}^2 & C_{n-1}^3 & \dots & C_{n-1}^{n-1} \\ 0 & 1 & C_{n-1}^1 & C_{n-1}^2 & \dots & C_{n-1}^{n-2} \\ 0 & 0 & 1 & C_{n-1}^1 & \dots & C_{n-1}^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

808. $\begin{pmatrix} 3197 & -1266 \\ 7385 & -922 \end{pmatrix}$.

809. $\begin{pmatrix} 190 & 189 & -189 \\ 126 & 127 & -126 \\ 252 & 252 & -251 \end{pmatrix}$.

811. (a) The ith and jth rows of the product are interchanged; (b) to the ith row of the product is added the jth row multiplied by c; (c) the ith and jth columns of the product are interchanged; (d) to the ith column of the product is added the jth column multiplied by c.

822. $\begin{pmatrix} a & 2b \\ 3b & a+3b \end{pmatrix}$, where a and b are arbitrary numbers.

823. $\begin{pmatrix} a & 3b \\ -5b & a+9b \end{pmatrix}$, where a and b are arbitrary numbers.

824. $\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$.

825. $\begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$.

826. $c=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$ $(k=0, 1, 2, \ldots, n-1)$, n is the order of the matrix A.

827. $f(A)=\begin{pmatrix} 21 & -23 & 15 \\ -13 & 4 & 10 \\ -9 & 22 & 25 \end{pmatrix}$.

828. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

830. *Hint.* Regard matrices of order n as n^2-dimensional vectors.

831. *Hint.* Apply problem 814.

Note. It is assumed in the problem that the elements of the matrices A and B are numbers. The result is incorrect for a field of characteristic $p \neq 0$. Thus, for matrices of order p,

$$A=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & p-1 & 0 \end{pmatrix},$$

we have $AB-BA=E$.

832. $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where a, b, c are arbitrary numbers that satisfy the relation $a^2+bc=0$.

833. *Hint.* Using problem 829, prove that if $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^k=(a+d)^{k-1}A$.

834. $\pm E$ or $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $a^2+bc=1$.

835. If $|A| \neq 0$, then $X=0$; if $|A|=0$, but $A \neq 0$, and the ratio of the elements of the first column of matrix A to the corresponding elements of the second column $=\alpha$, then $X=\begin{pmatrix} x & y \\ -\alpha x & -\alpha y \end{pmatrix}$ for arbitrary x and y; if both elements of the second column of A are zero, but at least one element of the first column is nonzero, then $X=\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}$ with arbitrary x and y; if $A=0$, then X is any matrix.

836. $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

837. $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$.

838. $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

839. $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$.

840. $\begin{pmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{pmatrix}$.

841. $\begin{pmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{pmatrix}$.

842. $\begin{pmatrix} -7/3 & 2 & -1/3 \\ 5/3 & -1 & -1/3 \\ -2 & 1 & 1\frac{1}{3} \end{pmatrix}$.

843. $1/9\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

844. $1/4\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

845. $\begin{pmatrix} 22 & -6 & -26 & 17 \\ -17 & 5 & 20 & -13 \\ -1 & 0 & 2 & -1 \\ 4 & -1 & -5 & 3 \end{pmatrix}$.

846. $\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

847. $\begin{pmatrix} 1 & -1 & 1 & -1 & \dots & (-1)^{n-1} \\ 0 & 1 & -1 & 1 & \dots & (-1)^{n-2} \\ 0 & 0 & 1 & -1 & \dots & (-1)^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$, n is the order of the given matrix.

848. $\begin{pmatrix} 1 & -a & 0 & 0 & \dots & 0 \\ 0 & 1 & -a & 0 & \dots & 0 \\ 0 & 0 & 1 & -a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$

849. $\begin{Vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a & 1 & 0 & 0 & \dots & 0 & 0 \\ a^2 & -a & 1 & 0 & \dots & 0 & 0 \\ -a^3 & a^3 & -a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (-a)^n & (-a)^{n-1} & (-a)^{n-2} & (-a)^{n-3} & \dots & -a & 1 \end{Vmatrix}$.

850. $$\begin{Vmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{Vmatrix}.$$

851. $$\frac{1}{n+1}\begin{Vmatrix} n & n-1 & n-2 & n-3 & \dots & 1 \\ n-1 & 2(n-1) & 2(n-2) & 2(n-3) & \dots & 2 \\ n-2 & 2(n-2) & 3(n-2) & 3(n-3) & \dots & 3 \\ n-3 & 2(n-3) & 3(n-3) & 4(n-3) & \dots & 4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & 4 & \dots & n \end{Vmatrix}.$$

852. $$\begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix}.$$

853. $$\frac{1}{n-1}\begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & 2-n & 1 & \dots & 1 \\ 1 & 1 & 2-n & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2-n \end{pmatrix}.$$

854. $$-\frac{1}{a(n+a)}\begin{pmatrix} 1-n-a & 1 & 1 & \dots & 1 \\ 1 & 1-n-a & 1 & \dots & 1 \\ 1 & 1 & 1-n-a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1-n-a \end{pmatrix}.$$

855. $$-\frac{1}{s}\begin{Vmatrix} \frac{1-a_1 s}{a_1^2} & \frac{1}{a_1 a_2} & \frac{1}{a_1 a_3} & \dots & \frac{1}{a_1 a_n} \\ \frac{1}{a_2 a_1} & \frac{1-a_2 s}{a_2^2} & \frac{1}{a_2 a_3} & \dots & \frac{1}{a_2 a_n} \\ \frac{1}{a_3 a_1} & \frac{1}{a_3 a_2} & \frac{1-a_3 s}{a_3^2} & \dots & \frac{1}{a_3 a_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{a_n a_1} & \frac{1}{a_n a_2} & \frac{1}{a_n a_3} & \dots & \frac{1-a_n s}{a_n^2} \end{Vmatrix},$$

where $s = 1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$.

857. $\begin{pmatrix} -7 & 5 & 12 & -19 \\ 3 & -2 & -5 & 8 \\ 41 & -30 & -69 & 111 \\ -59 & 43 & 99 & -159 \end{pmatrix}$

858. $\frac{1}{ns}\begin{pmatrix} 1-s & 1+s & 1 & \dots & 1 & 1 \\ 1 & 1-s & 1+s & \dots & 1 & 1 \\ 1 & 1 & 1-s & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1+s & 1 & 1 & \dots & 1 & 1-s \end{pmatrix}$, where $s=\frac{n(n+1)}{2}$.

Hint. In the system of equations, for elements of the kth column of the inverse matrix, subtract from each equation—from the first to the $(n-1)$th—the next equation and then add the resulting $n-1$ equations. Express all unknowns in terms of the kth.

859. $\frac{1}{nhs}\begin{pmatrix} h-s & h+s & h & \dots & h & h \\ h & h-s & h+s & \dots & h & h \\ h & h & h-s & \dots & h & h \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h+s & h & h & \dots & h & h-s \end{pmatrix}$,

where $s = na + h\cdot\frac{n(n-1)}{2}$ is the sum of the elements of some row (or column) of the given matrix.

860. $\frac{1}{n}\begin{Vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \varepsilon^{-3} & \dots & \varepsilon^{-(n-1)} \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \varepsilon^{-6} & \dots & \varepsilon^{-2(n-1)} \\ 1 & \varepsilon^{-3} & \varepsilon^{-6} & \varepsilon^{-9} & \dots & \varepsilon^{-3(n-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{-(n-1)} & \varepsilon^{-2(n-1)} & \varepsilon^{-3(n-1)} & \dots & \varepsilon^{-(n-1)^2} \end{Vmatrix}$.

Hint. Write the equations with unknowns $x_1, x_2, \dots, x_n$ in order to determine the elements of the kth column of the inverse matrix. Multiply each equation by a power of ε such that the coefficient of a definite unknown x_j becomes unity. Add the resulting equations.

861. $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$. 862. $\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$. 863. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

864. $\begin{pmatrix} 6 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix}$. 865. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. 866. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

867. The general aspect of the solution is $\begin{pmatrix} \frac{2+3c_1}{2} & \frac{3+3c_2}{2} \\ c_1 & c_2 \end{pmatrix}$, where c_1 and c_2 are arbitrary numbers.

868. The general aspect of the solution is $\begin{pmatrix} c_1 & \frac{2-3c_1}{4} \\ c_2 & \frac{9-3c_2}{4} \end{pmatrix}$, where c_1 and c_2 are arbitrary numbers.

869. There is no solution.

870. The general aspect of the solution is

$$\begin{pmatrix} 7-3c_1 & 5-3c_2 & 7-3c_3 \\ c_1 & c_2 & c_3 \\ 5c_1-9 & 5c_2-3 & 5c_3-7 \end{pmatrix},$$ where c_1, c_2, c_3 are arbitrary numbers.

871. $$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

872. In the matrix A^{-1}: (a) the ith and jth columns will be interchanged; (b) the ith column will be multiplied by $\frac{1}{c}$; (c) the ith column multiplied by c will be subtracted from the jth column.

When the columns of matrix A are transformed, the rows of matrix A^{-1} undergo a similar transformation.

879. $$A^{-1} = \begin{pmatrix} E_k & -U \\ 0 & E_l \end{pmatrix}.$$

881. When A is premultiplied by H_1, all rows of A are shifted one position upwards; the first row vanishes, and the last row is replaced by a zero row.

When A is postmultiplied by H_1, a similar alteration occurs with a shift of columns to the right. In the case of multiplication by H_{-1} on the left (or on the right) there is a similar shift of rows downwards (or of columns to the left).

890. The condition $AB = -BA$ is satisfied, for example, for the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Hint. When constructing the matrices A and B, take advantage of the hint in problem 1747.

897. *Hint.* Use the value of the adjugate determinant (problem 506) and reduce the problem to the preceding one.

898. *Hint.* Use the hint of the preceding problem.

899. *Hint.* Use the identity of problem 502 or the Binet-Cauchy formula in problem 499.

900. *Hint.* Use the identity of problem 504.

901. *Hint.* Use the Binet-Cauchy formula in problem 499.

902. *Hint.* Use the Binet-Cauchy formula in problem 499.

903. *Hint.* Use the problems 896 and 507.

904. *Hint.* Make use of problem 507.

905. The diagonal elements are equal to ± 1.

906. The diagonal elements are equal to unity in absolute value.

913. *Hint.* Make use of the Binet-Cauchy formula given in problem 499 or apply the hint given in that problem.

915. *Hint.* Take advantage of the preceding problem.

920. *Hint.* Take advantage of problem 913.

921. *Hint.* Use the Laplace theorem, the Cauchy-Bunyakovsky inequality and the Binet-Cauchy formula (see problems 503 and 499).

922. *Hint.* Let n be the number of rows of A, B, C; k the number of columns of B, and l the number of columns of C. Check that the inequality holds for $k + l > n$ and that the rank of $A < k + l \leqslant n$.

Show that for $k + l = n$ the problem coincides with the preceding one. Verify that the inequality becomes an equality when $k + l \leqslant n$ and under the supplemental condition $B' \cdot C = 0$. Finally, when the rank of $A = k + l < n$, complete A to a square matrix $(A, D) = P = (B, Q)$, where $Q = (C, D)$, with the aid of $n - k - l$ linearly independent columns so that $A'D = 0$ (this can be done by constructing a fundamental set of solutions of a homogeneous system of equations with matrix A'), apply the preceding case to the matrices $P = (A, D)$ and $Q = (C, D)$, and take into account that $P = (B, Q)$ and $|D'D| > 0$.

923. *Hint.* Apply the inequality of the preceding problem several times.

924. *Hint.* Repeatedly apply the inequalities of problem 922.

925. *Hint.* Reason as indicated in the hint to problem 922.

926. *Hint.* Make repeated use of the inequality of the preceding problem and take advantage of the answer to problem 532.

927. The ith and jth rows or the ith and jth columns are interchanged when multiplied by a matrix whose elements $p_{kk} = 1$ for k not equal to i, j; $p_{ij} = p_{ji} = 1$, and otherwise zero.

The ith row (or column) is multiplied by a number $c \neq 0$ when we have multiplication by a matrix that differs from the unit matrix solely in that the ith element of the principal diagonal a_{ii} is equal to c. Addition of the jth row multiplied by c to the ith row results when premultiplying by a matrix that differs from the unit matrix only in that the element $p_{ij} = c$.

For a similar transformation of columns, multiply by a similar matrix in which $p_{ji} = c$.

Hint. To determine the type of desired matrices, perform the given elementary transformation with respect to the unit matrix whose order is equal to the number of rows of matrix A in the case of a transformation of rows and to the number of columns of A in the case of a transformation of columns. Verify that the resulting matrices satisfy the requirements of the problem.

928. *Hint.* Use problem 927 and show that transformation of type (a) may be replaced by several transformations of types (b) and (c).

929. *Hint.* Take advantage of problems 617 and 927.

930. *Hint.* Use problem 623.

931. *Hint.* Apply problem 929 to matrix A and take advantage of problems 915, 930, and 914.

933. *Hint.* Use problem 927.

934. $\begin{pmatrix} 24 & 3 & -4 & -8 \\ -23/2 & -1 & 2 & 7/2 \\ 10 & 1 & -2 & -3 \\ -5 & 0 & 1 & 1 \end{pmatrix}$.

935. $\begin{pmatrix} -1 & 2 & -1 \\ 1/4 & -9/4 & 3/2 \\ 1/4 & 3/4 & -1/2 \end{pmatrix}$.

936. $\begin{pmatrix} -1/6 & 1/2 & -7/6 & 10/3 \\ -7/6 & -1/2 & 5/6 & -5/3 \\ 3/2 & 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \end{pmatrix}$.

938. *Hint.* To prove necessity, take for B any nonzero column of matrix A.

939. *Hint.* To prove necessity, take for B the matrix of any r linearly independent columns of matrix A; make up the ith column of matrix C from coefficients in the expression of the ith column of A in terms of the columns of B. Use problem 914.

941. *Hint.* Make use of the problems 626 and 931.

943. *Hint.* Use integral elementary transformations to bring the given matrix A to a normal matrix. To do this, choose the least (in absolute value) of the nonzero elements and then, via integral elementary transformations, replace the elements of the row and of the column in which this element appears by their remainders when divided by that element. Repeat this until all elements of the ith row and the jth column, except a_{ij}, vanish. If some element a_{kl} in the new matrix is not divisible by a_{ij}, then add the kth row to the ith row and again pass to the remainder. Repeat until all elements of some pth row and qth column, except a_{pq}, are zeros and all other elements are divisible by a_p. Then transfer a_{pq} to the upper left-hand corner and begin similar transformations with the reduced matrix obtained by deleting the first row and the first column, and so forth. To prove the uniqueness of the normal form, denote by d_k the largest common divisor of all elements of matrix A of rank r with m rows and n columns when $k = 1, 2, \ldots, r$ and set $d_k = 0$ when $r < k \leqslant m, n$. Show that the divisors of the minors d_k do not change under integral elementary transformations and that the elements $e_1, e_2, \ldots$ on the main diagonal of the normal matrix, which is equivalent to A, are connected with the divisors of the minors by the equalities $d_k = e_1, e_2, \ldots, e_k$ $(k \leqslant m, n)$, whence

$$e_k = \frac{d_k}{d_{k-1}} \quad (k = 1, 2, \ldots, r) \quad \text{and} \quad e_k = 0 \quad \text{for} \quad r < k \leqslant m, n.$$

944. *Hint.* When proving the possibility of the required representation, make use of problem 927. When proving the uniqueness of the two representations $A = P_1R_1 = P_2R_2$, derive that the matrix $C = P_2^{-1}P_1 = R_2R_1^{-1}$ is integral, unimodular, and triangular, whose elements on the main diagonal are positive and, hence, are equal to unity. Then by equating the $(k+1)$th, $(k+2)$th etc. elements of the kth row in the equality $CR_1 = R_2$, show that all elements

of matrix C to the right of the main diagonal are equal to zero, that is, $C = E$.

945. *Hint.* Make use of problem 927.

949. *Solution.* Reduce the matrix A to the upper triangular matrix C via the following elementary operations: $a_{11} = d_1 \neq 0$. Subtracting the first row multiplied by suitable numbers from the other rows, reduce A to the form

$$A^1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a^1_{22} & a^1_{23} & \dots & a^1_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a^1_{n2} & a^1_{n3} & \dots & a^1_{nn} \end{pmatrix};$$

since the minors involving the first row do not change in this process, we have $d_2 = a_{11}a^1_{22} \neq 0$, whence $a^1_{22} \neq 0$. Subtracting the second row of matrix A^1 with suitable factors from the rows below, make the second elements of these rows vanish, and so on. After r steps, all elements of the first r columns lying below the diagonal will vanish. Since the rank of the resulting matrix $A^{(r)} = C$ is equal to the rank of A, that is, equal to r, it follows that all elements of the last $n - r$ rows of matrix C are zero and, hence, C is an upper triangular matrix. By virtue of problem 927, $C = PA$, where P is a product of a series of lower triangular matrices, that is to say, again a lower triangular matrix $A = P^{-1}C = BC$, where $B = P^{-1}$ is a lower triangular matrix. This proves the existence of an expansion of the type (2).

Suppose we have any representation of the type (2). By the formula for minors of a product of matrices (problem 913) we have

$$A\begin{pmatrix} 1, & 2, & \dots, & k-1, & i \\ 1, & 2, & \dots, & k-1, & k \end{pmatrix} = \sum_{j_1<j_2<\dots<j_k} B\begin{pmatrix} 1, & 2, & \dots, & k-1, & i \\ j_1, & j_2, & \dots, & j_{k-1}, & j_k \end{pmatrix} C\begin{pmatrix} j_1, & j_2, & \dots, & j_{k-1}, & j_k \\ 1, & 2, & \dots, & k-1, & k \end{pmatrix}.$$

But the first k columns of matrix C contain only one kth-order minor different from zero, and for that reason

$$A\begin{pmatrix} 1, & 2, & \dots, & k-1, & i \\ 1, & 2, & \dots, & k-1, & k \end{pmatrix} = B\begin{pmatrix} 1, & 2, & \dots, & k-1, & i \\ 1, & 2, & \dots, & k-1, & k \end{pmatrix} C\begin{pmatrix} 1, & 2, & \dots, & k \\ 1, & 2, & \dots, & k \end{pmatrix}$$
$$= b_{11}b_{22} \dots b_{k-1,\,k-1}b_{ik}c_{11}c_{22} \dots c_{kk} \quad (i = k,\ k+1,\ \dots,\ n;\ k = 1,\ 2,\ \dots,\ r). \qquad \text{(a)}$$

Setting $i = k$ here, we get

$$d_k = b_{11}b_{22} \dots b_{kk}c_{11}c_{22} \dots c_{kk} \quad (k = 1,\ 2,\ \dots,\ r). \qquad \text{(b)}$$

Dividing (b) by a similar equality with k replaced by $k - 1$, we get (3). Dividing (a) by (b), we get the first of the formulas (4). The second formula is obtained in similar fashion.

Let D be any nonsingular diagonal matrix with elements $d_1, d_2, \dots, d_n$ on the main diagonal. Then $A = BC = (BD)(D^{-1}C)$. The matrix BD is obtained from B by multiplying the columns by $d_1, d_2, d_3, \dots, d_n$. The matrix $D^{-1}C$ is obtained from C by multi-

plying the rows by $D_1^{-1}, D_2^{-1}, \ldots, D_n^{-1}$. Therefore the diagonal elements of B and C may be taken arbitrarily, given the conditions (3). In the product BC, the elements of the last $n - r$ columns of B are multiplied by the elements of the last $n - r$ rows of C. For this reason, if some of the elements are put equal to zero, the others may be taken in arbitrary fashion.

951. *Hint.* Use the solution of problem 949 and show that under the conditions $b_{kk} = c_{kk} = \sqrt{\frac{d_k}{d_{k-1}}}$ the conditions (3) are satisfied and the conditions (4) yield $b_{ik} = c_{ki}$ $(i = k+1, k+2, \ldots, n, k = 1, 2, \ldots, r)$.

952. $AB = C = (C_{ij})$, where $C_{11} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$, $C_{12} = \begin{pmatrix} 2 & 4 \\ 9 & 6 \end{pmatrix}$, $C_{21} = (8)$, $C_{22} = (9, 1)$, $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 6 & 2 & 4 \\ 9 & 9 & 6 \\ 8 & 9 & 1 \end{pmatrix}$.

954. *Hint.* Consider the product of the ith block row by the ith block column.

957. P is a square block matrix with square unit blocks of orders $m_1, m_2, \ldots, m_s$ along the main diagonal, and the block lying at the intersection of the ith block row and the jth block column coincides with matrix X, while all other off-diagonal blocks are zero. Similarly, Q is a block matrix with square unit blocks of orders $n_1, n_2, \ldots, n_t$ on the main diagonal, with matrix Y at the intersection of the jth block row and the ith block column, and with zero blocks elsewhere.

958. *Hint.* From the second block row subtract the first multiplied on the left by the matrix CA^{-1} and, using the preceding problem, show that the rank remains unchanged in the process.

959. *Solution.* The same series of elementary operations that carries matrix R into R_1 carries the matrix $T = \begin{pmatrix} A & B \\ -C & -CA^{-1}B \end{pmatrix}$ into the matrix $T_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & X - CA^{-1}B \end{pmatrix}$. By the preceding problem, the rank of T is equal to n. Since elementary operations do not alter the rank of a matrix, the rank of T_1 = the rank of $T = n$ and the rank of A_1 = the rank of $A = n$. Therefore, $X - CA^{-1}B = 0$; $X = CA^{-1}B$.

960. $A^{-1} = \begin{pmatrix} -20 & 8 & 11 \\ 17 & -7 & -9 \\ -2 & 1 & 1 \end{pmatrix}$.

961. $x = 1$, $y = 2$, $z = 3$.

962. $X = \begin{pmatrix} 9 & 8 \\ 2 & 3 \end{pmatrix}$.

963. *Solution.* The properties (a) and (b) follow readily from the definition of a Kronecker product. To prove (c), we will write for the indices of the elements of the Kronecker product not the numbers of the pairs but the pairs themselves (and the numbers of a single pair will be written together without parentheses). Set $AB = F$,

$CD = G$, $F \times G = H$, $A \times C = P$, $B \times D = Q$. Then

$$h_{i_1 i_2,\, j_1 j_2} = f_{i_1, j_1} g_{i_2, j_2} = \sum_{s=1}^{m} a_{i_1, s} b_{s, j_1} \sum_{t=1}^{n} c_{i_2, t} d_{t, j_2}$$

$$= \sum_{s,\, t} a_{i_1, s} c_{i_2, t} b_{s, j_1} d_{t, j_2} = \sum_{s,\, t} p_{i_1 i_2,\, st} q_{st,\, j_1 j_2},$$

whence $H = PQ$.

964. (a) For the right direct product, take the lexicographic arrangement of pairs: (1, 1), (1, 2), . . ., (1, n), (2, 1), (2, 2), ..., (2, n), . . ., (m, n). For the left product, take the arrangement (1, 1), (2, 1), . . ., (m, 1), (1, 2), (2, 2), . . ., (m, 2), . . ., (m, n) that is obtained by lexicographic recording of the same pairs read from right to left. The properties (b), (c), and (d) follow directly from the definition. The property (e) follows from the relations (c) of the preceding problem and (d) of the present problem. The properties of the left product follow from the appropriate properties of the right product with the aid of relations (b).

965. *Hint.* Show that a change in the numbering of the pairs does not alter the determinant $|A \times B|$ and, using property (c) of problem 963, express it in the form $|A \times B| = |(AE_m) \times (E_n B)| = |A \times E_n| \cdot |E_m \times B| = |A^* \times E_n| \times |E_m \times {}^*B|$.

966. *Hint.* Use the proposition that from $|A| = 0$ follows the rank of $\hat{A} \leqslant 1$ that was proved in problem 630.

967. *Solution.* Set $AB = C$ and denote the cofactors by A_{ij}, B_{ij}, C_{ij} respectively, and the minors of the elements of the ith row and the jth column of matrices A, B, C by M_{ij}, N_{ij}, P_{ij} respectively.

Then, applying the expression for the minor of a product of two matrices in terms of the minors of the matrices (problem 913), we obtain

$$\hat{c}_{ij} = C_{ji} = (-1)^{j+i} P_{ji} = (-1)^{j+i} \sum_{k=1}^{n} M_{jk} N_{ki}$$

$$= \sum_{k=1}^{n} (-1)^{j+k} M_{jk} (-1)^{k+i} N_{ki} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} \hat{B}_{ik} \hat{A}_{kj},$$

whence $\hat{C} = \hat{B} \cdot \hat{A}$.

For the nonsingular matrices A and B, the same result is obtained more readily thus: by the preceding problem $\hat{A} = |A| \cdot A^{-1}$, whence $(\widehat{AB}) = |AB| (AB)^{-1} = |A| \cdot |B| B^{-1} A^{-1} = \hat{B}\hat{A}$. For matrices with numerical elements, the case of singular matrices is obtained by a passage to the limit. A polynomial in λ equal to the determinant $|A + \lambda E|$ has a degree of n and not more than n roots. And so we can take the sequence of numbers: $\lambda_1, \lambda_2, \ldots, \lim_{k \to \infty} \lambda_k = 0$ such that the matrices $A + \lambda_k E$ and $B + \lambda_k E$ are nonsingular. From what has been proved, $[(A + \lambda_k E)(B + \lambda_k E)]\hat{} = (B + \lambda_k E)\hat{} \cdot (A + \lambda_k E)\hat{}$. Passing to the limit as $k \to \infty$, we obtain $(\widehat{AB}) = \hat{B}\hat{A}$.

968. *Hint.* (a) This follows from problem 913; (b) prove for the case where $|A| = 0$ using problem 747, and for $|A| \neq 0$ using the relation $\hat{A} = |A| \cdot CA'^{-1}C$, where C is a diagonal matrix with elements 1, —1, 1, —1 . . . on the main diagonal.

969. *Hint.* Use problem 913.

970. For example, the numbering of the combinations is in lexicographic order in which the combination $i_1 < i_2 < \ldots < i_p$ precedes the combination $j_1 < j_2 < \ldots < j_p$ if the first nonzero difference $j_1 - i_1, j_2 - i_2, \ldots, j_p - i_p$ is positive.

971. *Hint.* Prove the proposed equality first for a triangular matrix A by using the fact that changing the numbering order of the combinations does not change the determinant of the associated matrix A_p and also by taking advantage of the preceding problem. Reduce the general case to triangular matrices with the aid of problems 928 and 969.

972. *Solution.* By virtue of problem 956, from $AB = E_n$ follows $A_pB_p = E_N$, where $N = C_n^p$, whence

$$\sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} A \begin{pmatrix} j_1, & j_2, & \ldots, & j_p \\ i_1, & i_2, & \ldots, & i_p \end{pmatrix} B \begin{pmatrix} i_1, & i_2, & \ldots, & i_p \\ k_1, & k_2, & \ldots, & k_p \end{pmatrix}$$

$$= \begin{cases} 1, & \text{if } \sum_{s=1}^{p} (j_s - k_s)^2 = 0, \\ 0, & \text{if } \sum_{s=1}^{p} (j_s - k_s)^2 > 0 \end{cases} \quad \left(1 \leqslant \begin{matrix} j_1 < j_2 < \ldots < j_p \\ k_1 < k_2 < \ldots < k_p \end{matrix} \leqslant n\right). \tag{2}$$

On the other hand, by the Laplace theorem we find

$$\sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} A \begin{pmatrix} j_1, & j_2, & \ldots, & j_p \\ i_1, & i_2, & \ldots, & i_p \end{pmatrix} (-1)^{\sum_{s=1}^{p} (i_s + k_s)} \times A \begin{pmatrix} k_1', & k_2', & \ldots, & k_{n-p}' \\ i_1', & i_2', & \ldots, & i_{n-p}' \end{pmatrix}$$

$$= \begin{cases} |A|, & \text{if } \sum_{s=1}^{p} (j_s - k_s)^2 = 0, \\ 0, & \text{if } \sum_{s=1}^{p} (j_s - k_s)^2 > 0, \end{cases} \tag{3}$$

where $i_1' < i_2' < \ldots < i_{n-p}'$ together with $i_1 < i_2 < \ldots < i_p$ and $k_1' < k_2' < \ldots < k_{n-p}'$ together with $k_1 < k_2 < \ldots < k_p$ constitute a complete system of indices $1, 2, \ldots, n$.

Since the system of linear equations with the nonsingular matrix A_p for the given constant terms has a unique solution and since the right members of equation (3) differ from the right members of the corresponding equalities (2) in the factor $|A|$ alone, then so also should the left-hand members differ in that factor, whence follow the required equations (1).

973. *Hint.* Use the Laplace theorem and problem 903.

974. *Hint.* Use the Laplace theorem and problem 903.

975. $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^3 \end{pmatrix}$. 976. $\begin{pmatrix} \lambda+1 & 0 \\ 0 & \lambda^3+\lambda^2-2\lambda-2 \end{pmatrix}$.

977. $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^2+5\lambda \end{pmatrix}$. 978. $\begin{pmatrix} \lambda-1 & 0 \\ 0 & (\lambda+1)(\lambda-1)^3 \end{pmatrix}$.

979. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$. 980. $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^2+\lambda & 0 \\ 0 & 0 & \lambda^3+\lambda^2 \end{pmatrix}$.

981. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\lambda-2)^3 \end{pmatrix}$. 982. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^2+\lambda & 0 \\ 0 & 0 & \lambda^3+2\lambda^2+\lambda \end{pmatrix}$.

983. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2+\lambda \end{pmatrix}$.

984. *Hint.* Prove that the polynomials $D_k(\lambda)$ do not change under elementary operations and that in the case of a normal diagonal form, $D_k(\lambda) = \prod_{i=1}^{k} E_i(\lambda)$ $(k = 1, 2, \ldots, n)$.

985. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda(\lambda-1)(\lambda-2) & 0 \\ 0 & 0 & \lambda(\lambda-1)(\lambda-2) \end{pmatrix}$.

986. $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda(\lambda-1)(\lambda-2)(\lambda-3) \end{pmatrix}$.

987. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, where $p=(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$.

988. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}$, where p is the product of polynomials a, b, c, d divided by the product of the leading coefficients of the polynomials.

989. $\begin{pmatrix} d(\lambda) & 0 \\ 0 & \dfrac{f(\lambda)\,g(\lambda)}{cd(\lambda)} \end{pmatrix}$, where $d(\lambda)$ is the largest common divisor of the polynomials $f(\lambda)$ and $g(\lambda)$ having leading coefficient equal to unity and having c as the product of the leading coefficients of the polynomials.

990. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & fgh & 0 \\ 0 & 0 & fgh \end{pmatrix}$.

991. $\begin{pmatrix} abc & 0 & 0 \\ 0 & \frac{fgh}{abc} & 0 \\ 0 & 0 & fgh \end{pmatrix}$.

where a, b, c are respectively the largest common divisors of g and h, f and h, f and g taken with leading coefficients equal to unity.

992. $\begin{pmatrix} \frac{abc}{d} & 0 & 0 \\ 0 & \frac{d^2fgh}{abc} & 0 \\ 0 & 0 & \frac{fgh}{d} \end{pmatrix}$,

where d is the largest common divisor of f, g and h; and a, b, c are respectively the largest common divisors of g and h, f and h, and f and g, the leading coefficients of all polynomials a, b, c, d being unity.

993. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^4 \end{pmatrix}$.

994. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}$.

995. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & f(\lambda) \end{pmatrix}$, where $f(\lambda)=\lambda^5+5\lambda^4+4\lambda^3+3\lambda^2+2\lambda+1$.

996. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & [(\lambda+\alpha)^2+\beta^2]^2 \end{pmatrix}$, if $\beta \neq 0$, or

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\lambda+\alpha)^2 & 0 \\ 0 & 0 & 0 & (\lambda+\alpha)^2 \end{pmatrix}$, if $\beta=0$.

997. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & (\lambda-3)^2 \end{pmatrix}$.

998. $\begin{pmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & (\lambda-1)^2 \end{pmatrix}$.

999. $\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & \lambda^n \end{pmatrix}$. when n is the order of the given matrix.

1000. Equivalent. 1001. They are not equivalent.

1002. The matrices A and C are equivalent, but they are not equivalent to matrix B. 1003. The unit matrix.

1005. *Hint.* Take advantage of the fact that an elementary transformation of the rows of matrix A reduces to a premultiplication of A (a postmultiplication of the columns) by a special unimodular λ-matrix. Furthermore, if $B = P_s P_{s-1} \dots P_1 A Q_1 Q_2 \dots Q_t$ where P_i, Q_j are special unimodular λ-matrices, then set $P = P_s P_{s-1} \dots \dots P_1 E_m$ and $Q = E_n Q_1 Q_2 \dots Q_t$. When proving sufficiency, make use of the answer to problem 1003.

1006. $B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2-1 \end{pmatrix}$; $P = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & \lambda+3 \end{pmatrix}$;

$$Q = \begin{pmatrix} \lambda & -\frac{1}{2}\lambda^2+\frac{1}{2}\lambda-1 \\ 1-\lambda & \frac{1}{2}\lambda^2-\lambda+\frac{3}{2} \end{pmatrix}.$$

1007. $B = \begin{pmatrix} \lambda+2 & 0 \\ 0 & \lambda^2+4\lambda+4 \end{pmatrix}$; $P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$;

$$Q = \begin{pmatrix} 1 & -\lambda^2-2\lambda \\ -\lambda & \lambda^3+2\lambda^2+1 \end{pmatrix}.$$

1008. $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda^2-\lambda^2 \end{pmatrix}$; $P = \begin{pmatrix} -2\lambda-3 & 0 & 2\lambda+4 \\ -3 & -1 & 4 \\ 1 & 0 & -1 \end{pmatrix}$;

$$Q = \begin{pmatrix} 1 & -\lambda^3+\lambda^2-\lambda+1 & 0 \\ \lambda & -\lambda^4+\lambda^3-\lambda^2+\lambda+1 & 0 \\ -\lambda-1 & \lambda^4-2 & 1 \end{pmatrix}.$$

1009. For example,

$$P = \begin{pmatrix} -3\lambda^2+9\lambda+8 & 3\lambda^2-9\lambda+4 \\ -2\lambda^2+6\lambda+5 & 2\lambda^2-6\lambda+3 \end{pmatrix}; \quad Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

1010. For example,

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \quad Q = \begin{pmatrix} -\lambda^2+\lambda+1 & -2\lambda^2+\lambda+2 \\ \lambda-1 & 2\lambda-1 \end{pmatrix}.$$

1011. For example,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 11\lambda+8 & -11 & 0 \\ 2\lambda-1 & -2 & 1 \end{pmatrix}; \quad Q = \begin{pmatrix} -2\lambda-1 & -\lambda-1 & -\lambda^2+\lambda \\ 2\lambda^2+\lambda+2 & \lambda^2+\lambda+1 & \lambda^3-\lambda^2-1 \\ 0 & 0 & 1 \end{pmatrix}.$$

1012. For example,

$$P=\begin{pmatrix} 1 & 0 & 0 \\ -3\lambda^2-2\lambda+1 & \lambda^2+\lambda & 1 \\ -6\lambda^2-4\lambda+4 & 2\lambda^2+2\lambda-1 & 2 \end{pmatrix};$$

$$Q=\begin{pmatrix} -\lambda+1 & -\lambda & -\lambda^2 \\ -\lambda+2 & -\lambda+1 & -\lambda^2 \\ \lambda-1 & \lambda & \lambda^2+1 \end{pmatrix}.$$

1013. For example, $P=\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$; $Q=\begin{pmatrix} -3 & -5 & 0 \\ 5 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

1014. For example, $P=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $Q=\begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$.

1015. $E_1(\lambda)=1$; $E_2(\lambda)=\lambda-1$; $E_3(\lambda)=(\lambda-1)(\lambda^2-1)$.

1016. $E_1(\lambda)=\lambda+1$; $E_2(\lambda)=\lambda^2-1$; $E_3(\lambda)=\lambda^3-\lambda$.

1017. $E_1(\lambda)=\lambda^2+1$; $E_2(\lambda)=\lambda^3-\lambda^2+\lambda-1$; $E_3(\lambda)=E_4(\lambda)=0$.

1018. $E_1(\lambda)=1$; $E_2(\lambda)=\lambda^2-\lambda+1$; $E_3(\lambda)=\lambda^3+1$; $E_4(\lambda)=0$.

1019. $E_1(\lambda)=\ldots=E_n(\lambda)=1$; $E_{n+1}(\lambda)=\lambda^{n+1}$.

1020. $E_1(\lambda)=\ldots=E_{n-1}(\lambda)=1$; $E_n(\lambda)=(\lambda-\alpha)^n$ if $\beta\neq 0$, $E_1(\lambda)=\ldots=E_n(\lambda)=\lambda-\alpha$ if $\beta=0$.

Hint. When $\beta\neq 0$, show that the divisor of the minors $D_{n-1}(\lambda)=1$. To do this, check to see that the minor obtained by deleting the first column and the last row does not vanish when $\lambda=\alpha$.

1021. $\lambda+1$, $(\lambda-1)^2$. **1022.** $\lambda+1$, $\lambda-1$, $\lambda-1$.

1023. There are no elementary divisors.

1024. $\lambda+1$, $(\lambda-1)^2$, $\lambda-1$, $\lambda+2$, $\lambda+2$, $\lambda+2$.

1025. There are no elementary divisors.

1026. In the field of rational numbers: λ^2+1, λ^2-3; in the field of real numbers: λ^2+1, $\lambda+\sqrt{3}$, $\lambda-\sqrt{3}$; in the field of complex numbers: $\lambda+i$, $\lambda-i$, $\lambda+\sqrt{3}$, $\lambda-\sqrt{3}$.

1027. In the field of rational numbers: λ^2-2, $(\lambda^2+4)^2$, λ^2+4; in the field of real numbers: $\lambda+\sqrt{2}$, $\lambda-\sqrt{2}$, $(\lambda^2+4)^2$, λ^2+4; in the field of complex numbers: $\lambda+\sqrt{2}$, $\lambda-\sqrt{2}$, $(\lambda+2i)^2$ $(\lambda-2i)^2$, $\lambda+2i$, $\lambda-2i$.

1028. In the field of rational numbers and in the field of real numbers: $(\lambda-1)^2$, $(\lambda+1)^2$, $\lambda+1$, $(\lambda^2-\lambda+1)^2$, $\lambda^2-\lambda+1$; in the field of complex numbers:

$$(\lambda-1)^2,\ (\lambda+1)^2,\ \lambda+1,\ \left(\lambda-\frac{1+i\sqrt{3}}{2}\right)^2,\ \left(\lambda-\frac{1-i\sqrt{3}}{2}\right)^2,$$

$$\lambda-\frac{1+i\sqrt{3}}{2},\quad \lambda-\frac{1-i\sqrt{3}}{2}.$$

1029. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 & 0 \\ 0 & 0 & 0 & \lambda^4-2\lambda^2+1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

1030. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda^3+2\lambda^2-4\lambda-8 & 0 \\ 0 & 0 & 0 & \lambda^6-12\lambda^4+48\lambda^2-64 \end{pmatrix}$.

1031. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda-2 & 0 & 0 \\ 0 & 0 & 0 & \lambda^5+\lambda^4-5\lambda^3-\lambda^2+8\lambda-4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

1032. *Hint.* Suppose $e(\lambda)$ is some kind of irreducible factor in the decomposition of at least one diagonal element, let s be the number of diagonal elements different from zero, and let $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \ldots \leqslant \alpha_s$ be the collection of exponents which $e(\lambda)$ encounters in these elements. Show that for $k = 1, 2, \ldots, s$ the divisor of the minors $D_k(\lambda)$ is exactly divisible by $[e(\lambda)]^{\alpha_1+\alpha_2+\ldots+\alpha_k}$ and the invariant factor $E_k(\lambda)$ is exactly divisible by $[e(\lambda)]^{\alpha_k}$.

1033. *Hint.* Use elementary operations to bring each diagonal block to diagonal (for example, normal) form and take advantage of the preceding problem.

1034. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda(\lambda+1) & 0 & 0 \\ 0 & 0 & \lambda(\lambda+1)(\lambda-1) & 0 \\ 0 & 0 & 0 & \lambda^2(\lambda+1)^2(\lambda-1)^2 \end{pmatrix}$.

1035. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^2-4\lambda & 0 & 0 \\ 0 & 0 & \lambda^2-4\lambda & 0 \\ 0 & 0 & 0 & \lambda^4-4\lambda^2 \end{pmatrix}$.

1036. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f(\lambda) & 0 & 0 \\ 0 & 0 & f(\lambda) & 0 \\ 0 & 0 & 0 & [f(\lambda)]^2 \end{pmatrix}$, where $f(\lambda) = \lambda^3+\lambda^2-6\lambda$.

1037. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^2-1 & 0 & 0 \\ 0 & 0 & (\lambda^2-1)^2 & 0 \\ 0 & 0 & 0 & (\lambda^2-1)^3 \end{pmatrix}$.

1038. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1039. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2-4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1040. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda-6 & 0 \\ 0 & 0 & 0 & \lambda^2+\lambda-6 \end{pmatrix}$.

1041. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2-1 & 0 & 0 \\ 0 & 0 & 0 & \lambda^4-2\lambda^2+1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

1042. $D_1 = 1$, $D_2 = 2$, $D_3 = 4$, $D_4 = 320$.

1043. $D_1 = 3$, $D_2 = 18$, $D_3 = 324$, $D_4 = 11\,664$.

1045. *Hint.* Make use of the preceding problem when proving the existence of representation of the given type. When proving uniqueness of the two representations of the given type $A = P_1R_1 = P_2R_2$, derive that the matrix $C = P_2^{-1}P_1 = R_2R_1^{-1}$ is a unimodular and triangular λ-matrix whose elements on the main diagonal have leading coefficient unity, which means that they themselves are equal to unity. Then, equating the elements of the kth row in the equation $CR_1 = R_2$ and taking into account the condition for the exponents of the elements R_1 and R_2, show that all elements of matrix C to the right of the main diagonal are equal to zero, that is, C is a unit matrix. From this obtain the equalities $P_1 = P_2$ and $R_1 = R_2$.

1047. For example, $A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$; $B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.

1048. Scalar matrices. *Hint.* Show that if $AT = TA$ for any nonsingular matrix T, then it holds true for all matrices T. To do this, represent the singular matrix T as $T = (T - \alpha E) + \alpha E$ where $\alpha \neq 0$ and is chosen so that $|T - \alpha E| \neq 0$. Then apply problem 818.

Note. This method of solution may prove inadequate for matrices with elements taken from a finite field where there may be an element α with the needed properties. A method suitable for matrices with elements taken from any field and not making use of problem 818 consists in the following: for $i \neq j$, denote by F_{ij} a matrix different from the unit merely in that the element in the ith row and jth column is equal to unity. In the equation $AF_{ij} = F_{ij}A$ we equate elements in the ith row and jth column and then in the ith row and ith column.

1049. For the matrix T, one can take the matrix obtained from the unit matrix by interchanging the ith and jth rows.

1050. *Hint.* Take advantage of the preceding problem.

1058. $A=(B-\lambda E)\begin{pmatrix}2\lambda+1 & \lambda & 1\\ 3\lambda+2 & 3\lambda & 2\\ \lambda+2 & 2\lambda & 1\end{pmatrix}+\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{pmatrix}$.

1059. $A=\begin{pmatrix}\lambda^2 & \lambda & 1\\ 2\lambda^2 & 3\lambda & 2\\ \lambda^2 & \lambda & -2\end{pmatrix}(B-\lambda E)+\begin{pmatrix}5 & -2 & 3\\ 6 & -3 & 2\\ 7 & -5 & 4\end{pmatrix}$.

1060. *Hint.* Use problem 1005.

1061. *Solution.* Suppose $P=(B-\lambda E)P_1+P_0$ and $Q=Q_1(B-\lambda E)+Q_0$. Using these relations, reduce

$$B-\lambda E=P(A-\lambda E)Q \tag{1}$$

to the form

$$B-\lambda E-P_0(A-\lambda E)Q_0=P(A-\lambda E)Q_1(B-\lambda E)$$
$$+(B-\lambda E)P_1(A-\lambda E)Q-(B-\lambda E)P_1(A-\lambda E)Q_1(B-\lambda E).$$

Substituting

$$P(A-\lambda E)=(B-\lambda E)Q^{-1} \text{ and } (A-\lambda E)Q=P^{-1}(B-\lambda E)$$

into this equation on the basis of (1) we get

$$B-\lambda E-P_0(A-\lambda E)Q_0=$$
$$=(B-\lambda E)[P_1P^{-1}+Q^{-1}Q_1-P_1(A-\lambda E)Q_1](B-\lambda E).$$

The expression in square brackets in the right-hand member of the equation must be zero because otherwise the right-hand member would have a power in λ not less than two, whereas the power of the left-hand member does not exceed unity. Therefore, $B-\lambda E=P_0(A-\lambda E)Q_0$. Equating the coefficients of λ and the constant terms in this equation, we get: $P_0Q=E$ and $B=P_0AQ_0$.

1063. Similar. 1064. Similar. 1065. The matrices A and C are similar, but they are not similar to B.

1066. The matrices B and C are similar, but they are not similar to matrix A.

1067. For example, $T=\begin{pmatrix}1 & -2\\ -1 & 3\end{pmatrix}$. *Hint.* To obtain the simplest possible answer, strive towards the simplest elementary operations on the columns of the matrices $A-\lambda E$ and $B-\lambda E$.

1068. For example, $T=\begin{pmatrix}0 & 2\\ -1 & 10\end{pmatrix}$. *Hint.* To reduce the matrix $A-\lambda E$ to normal diagonal form, from the second row mutiplied by 6 subtract the first row multiplied by $\lambda+16$, and from 6 times the first column subtract the second column multiplied by $\lambda-17$. Transform the matrix $B-\lambda E$ in similar fashion.

1069. For example,

$$T=\begin{pmatrix}1 & -3 & 3\\ 2 & -3 & 1\\ 1 & -2 & 1\end{pmatrix}.$$

1070. *Hint.* Obtain c_k as a sum of all factors in the determinant $|A - \lambda E|$ that appear in products of k elements each of the main diagonal and taken when $\lambda = 0$.

1071. $\lambda_1 = a_1^2 + a_2^2 + \ldots + a_n^2$, $\lambda_2 = \ldots = \lambda_n = 0$. *Hint.* Apply the preceding problem.

1074. *Hint.* Apply problem 1070 to the matrix $B = A - \lambda_0 E$ and show that the characteristic polynomial $|B - \mu E|$ of matrix B, after the substitution $\mu = \lambda - \lambda_0$, passes into the characteristic polynomial $|A - \lambda E|$ of matrix A.

1075. For a triangular matrix of the form

$$A = \begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & \lambda_0 & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_0 \end{pmatrix},$$

where $a_{i,\,i+1} \neq 0$ $(i = 1, 2, \ldots, n-1)$, we have $d = 1$. For a diagonal matrix of order n in which p elements of the main diagonal are equal to λ_0 we have $d = p$.

1076. *Hint.* Prove that

$$|A^{-1} - \lambda E| = (-\lambda)^n |A^{-1}| \cdot \left| A - \frac{1}{\lambda} E \right|.$$

1077. *Hint.* Multiply together the equalities

$$|A - \lambda E| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda),$$
$$|A + \lambda E| = (\lambda_1 + \lambda)(\lambda_2 + \lambda) \ldots (\lambda_n + \lambda)$$

and replace λ^2 by λ.

1078. *Hint.* Multiply together the equality $|A - \lambda E| = (\lambda_1 - \lambda) \times (\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$ and all equalities obtained from it by replacing λ by $\lambda\varepsilon_1$, $\lambda\varepsilon_2$, ..., $\lambda\varepsilon_{p-1}$, where $\varepsilon_k = \cos\frac{2\pi k}{p} + i \sin\frac{2\pi k}{p}$ $(k = 1, 2, \ldots, p-1)$, and in the resulting equality substitute λ for λ^p.

1079. *Solution.* Let $f(\lambda) = a_0 \prod_{j=1}^{s} (\lambda - \mu_j)$ and, besides, $\varphi(\lambda) = \prod_{i=1}^{n} (\lambda_i - \lambda)$. Setting $\lambda = A$ in $f(\lambda)$ we get: $f(A) = a_0 \prod_{j=1}^{s} (A - \mu_j E)$. Passing from matrices to determinants, we obtain

$$|f(A)| = a_0^n \prod_{j=1}^{s} |A - \mu_j E| = a_0^n \prod_{j=1}^{s} \varphi(\mu_j) = a_0^n \prod_{j=1}^{s} \prod_{i=1}^{n} (\lambda_i - \mu_j)$$
$$= \prod_{i=1}^{n} \left[a_0 \prod_{j=1}^{s} (\lambda_i - \mu_j) \right] = \prod_{i=1}^{n} f(\lambda_i).$$

On the other hand, $|f(A)| = a_0^n \prod_{j=1}^{s} \varphi(\mu_j) = R(f, \varphi)$.

1080. *Hint.* Apply the equality of the preceding problem to the polynomial $g(x) = f(x) - \lambda$, where λ is an arbitrary number.

1081. *Hint.* Apply the equality $|f(A)| = \frac{|g(A)|}{|h(A)|}$ and make use of problem 1079 and 1080.

1082. *Hint.* If at least one of the matrices A and B is nonsingular, then the assertion follows from the similarity of the matrices AB and BA (see problem 1047). In the general case, problems 920 and 1070 may be used. For matrices over a field with an infinite (or sufficiently large) number of elements, fulfilment of the required equality for nonsingular matrices implies that it is identically fulfilled. Finally, for matrices with numerical elements, the equality for a singular matrix A may be obtained by a passage to the limit. For example, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of the singular matrix A, then we take a sequence of numbers $\varepsilon_1, \varepsilon_2, \ldots$, such that all of them are different from $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lim\limits_{k\to\infty} \varepsilon_k = 0$. The matrix $A_k = A - \varepsilon_k E$ is nonsingular. Hence, $|A_k B - \lambda E| = |B A_k - \lambda E|$. Passing to the limit as $k \to \infty$, we obtain the needed equality.

1083. The eigenvalues (with account taken of mutiplicity) are: $\lambda_k = f(\varepsilon_k)$, where $f(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}$ and $\varepsilon_k = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n}$ $(k = 0, 1, 2, \ldots, n-1)$. *Hint.* Apply the expression for the circulant of problem 479 to the circulant $|A - \lambda E|$, where λ is a parameter.

1084. *Solution.* Applying problem 304 to the determinant $|A - \lambda E|$, where λ is the eigenvalue, set $\alpha + \beta = -\lambda$, $\alpha\beta = -1$. Then $|A - \lambda E| = \alpha^n + \alpha^{n-1}\beta + \ldots + \beta^n$. From $\alpha\beta = -1$ we find $\alpha \neq 0$ and $\beta \neq 0$. Furthermore, $\alpha \neq \beta$, since from $\alpha = \beta$ and $|A - \lambda E| = 0$ it should follow that $\alpha = \beta = 0$. Therefore, $|A - \lambda E| = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = 0$, whence $\left(\frac{\alpha}{\beta}\right)^{n+1} = 1$; $\frac{\alpha}{\beta} = \cos\frac{2\pi k}{n+1} + i \sin\frac{2\pi k}{n+1}$ $(k = 1, 2, \ldots, n)$, $k \neq 0$, since $\frac{\alpha}{\beta} \neq 1$. Solving this equation simultaneously with $\alpha\beta = -1$, we find $\alpha = \pm i\left(\cos\frac{\pi k}{n+1} + i \sin\frac{\pi k}{n+1}\right)$, $\beta = \pm i\left(\cos\frac{\pi k}{n+1} - i \sin\frac{\pi k}{n+1}\right)$. Here the signs $\pm$ must be taken coincident for α and β, since $\alpha\beta = -1$, whence $\lambda = -(\alpha + \beta) = \mp 2i \cos\frac{\pi k}{n+1}$ $(k = 1, 2, \ldots, n)$. All these must be eigenvalues. But among them there are some that are equal since $\cos\frac{\pi k}{n+1} = -\cos\frac{(n+1-k)\pi}{n+1}$.

All distinct eigenvalues are contained in the system

$$\lambda_k = 2i \cos\frac{\pi k}{n+1} \quad (k = 1, 2, \ldots, n).$$

But the degree of the characteristic polynomial is equal to n. Thus the latter system contains all eigenvalues, and there are no multiple roots.

1085. *Hint.* Prove that the Jordan submatrix of order k with the number α on the diagonal has a unique elementary divisor $(\lambda - \alpha)^k$.

Construct the Jordan matrix A_J whose Jordan submatrices are related as indicated with the elementary divisors of the matrix $A - \lambda E$ and, using problems 1033 and 1061, prove that the matrices A and A_J are similar. When proving uniqueness, use problem 1005 to make sure that the characteristic matrices of two similar Jordan submatrices B and C are equivalent and, using the coincidence of the elementary divisors of the matrices $B - \lambda E$ and $C - \lambda E$, again apply problem 1033 to make sure that the matrices B and C coincide to within the order of the blocks.

1086. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$.

1087. $\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$.

1088. The problem has not been posed properly. No such invariant factors are possible in a matrix $A - \lambda E$ of the fourth order.

1090. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

1091. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$.

1092. $\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$.

1093. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

1094. $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$.

1095. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

1096. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

1097. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

1098. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

1099. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

1100. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$.

1101. $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$.

1102. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{pmatrix}$. **1103.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}$, where $\varepsilon = -\frac{1}{2} \pm \pm \frac{1}{2} i\sqrt{3}$ is one of the complex values of $\sqrt[3]{1}$.

1104. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+3i & 0 \\ 0 & 0 & 2-3i \end{pmatrix}$. **1105.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

1106. $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. **1107.** $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

1108. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. **1109.** $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

1110. The same as in problem 1109. **1111.** The same as in problem 1109.

1112. $\begin{pmatrix} n & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & n & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & n & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & n & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & n \end{pmatrix}$. **1113.** $\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}$.

1114. $\begin{pmatrix} \alpha_0 & 0 & 0 & \dots & 0 \\ 0 & \alpha_1 & 0 & \dots & 0 \\ 0 & 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_{n-1} \end{pmatrix}$, where $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are all the values of $\sqrt[n]{\alpha^n}$, that is, $\alpha_k = \alpha\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)$ $(k=0, 1, 2, \dots, n-1)$.

1115. One Jordan submatrix with the number α on the main diagonal.

1118. In the field of rational numbers it is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

1119. In the field of real numbers it is similar to the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{pmatrix}.$$

1120. In the field of complex numbers it is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+3i & 0 \\ 0 & 0 & 2-3i \end{pmatrix}.$$

1121. It is not similar to the diagonal matrix in any field.

1125. The diagonal matrix with elements on the main diagonal equal to zero or to unity.

1126. A diagonal matrix with elements ± 1 on the main diagonal.

Note. The assertion does not hold true for matrices over a field of characteristic 2. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = E$ and matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ cannot be reduced to diagonal form due to the uniqueness of the Jordan form.

1127. If n is the period of the matrix A, that is, the smallest of the natural numbers k for which $A^k = E$, then on the main diagonal the diagonal matrix has certain of the n values of the root $\sqrt[n]{1}$.

Note. The result is not correct; for matrices over fields of a finite characteristic, for instance, for a matrix of order $\leqslant p$ over a field of characteristic p,

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

and the equality $A^p = E$ holds true.

1128. (a) $\lambda - 1$. (b) λ.

1129. For the scalar matrices $A = \alpha E$ and for them alone. There is only one such matrix for the given order n.

1130. $(\lambda - \alpha)^n$. **1134.** $\lambda^2 - 4\lambda + 4$. **1135.** $\lambda^2 - 5\lambda + 6$.

1136. For example, for the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\varphi(\lambda) = (\lambda - 1)^4$, $\psi(\lambda) = (\lambda - 1)^2$ but these matrices are not similar due to the uniqueness of the Jordan form of any matrix.

1137.

$$\begin{pmatrix} \alpha^k & C_k^1\alpha^{k-1} & C_k^2\alpha^{k-2} & C_k^3\alpha^{k-3} & \dots & C_k^{n-1}\alpha^{k-n+1} \\ 0 & \alpha^k & C_k^1\alpha^{k-1} & C_k^2\alpha^{k-2} & \dots & C_k^{n-2}\alpha^{k-n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & \alpha^k \end{pmatrix};$$

for $k \leqslant n - 1$ it is necessary to put $C_k^0 = 1$, and $C_k^s = 0$ for $k < s$.

1138. *Hint.* Set $A = \alpha E + H$ and in the equality

$$f(x) = f(\alpha) + \frac{f'(\alpha)}{1!}(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \ldots + \frac{f^{(s)}(\alpha)}{s!}(x-\alpha)^s$$

[s is the degree of the polynomial $f(x)$] set $x = A$.

1140. One Jordan submatrix with the number α^2 on the diagonal.

1141. If $n > 1$ is the order of the Jordan submatrix A with zero on the diagonal, then the Jordan form of the matrix A^2 consists of two blocks with zero on the diagonal, the blocks having the order $\frac{n}{2}$ for even n and the orders $\frac{n-1}{2}$, $\frac{n+1}{2}$ for odd n.

Hint. Use problem 1130 to find the minimal polynomials of the matrices A and A^2 and show that the blocks of the Jordan form of the matrix A^2 have orders that do not exceed $\frac{n}{2}$ for even n and $\frac{n+1}{2}$ for odd n. Verify that the divisor of the minors $D_{n-2}(\lambda)$ of the matrix $A^2 - \lambda E$ is equal to unity and then show what the Jordan form of the matrix A^2 contains at most two blocks.

1143. The desired matrix contains two Jordan submatrices with the number α on the diagonal, the orders of the submatrices being $\frac{n}{2}$ for even n or $\frac{n-1}{2}$ and $\frac{n+1}{2}$ for odd n. *Hint.* Use the two preceding problems.

1144. *Solution.* Let $A = TBT^{-1}$, where A is the given matrix and

$$B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}$$

is its Jordan form with the Jordan submatrices

$$B_i = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_i \end{pmatrix} \quad (i = 1, 2, \ldots, k).$$

Then $A' = T'^{-1}B'T'$. Let

$$H_i = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & 0 & 0 \end{pmatrix}$$

let it have the same order as B_i and let

$$H = \begin{pmatrix} H_1 & & & 0 \\ & H_2 & & \\ & & \ddots & \\ 0 & & & H_k \end{pmatrix}.$$

Direct multiplication yields: $B_i' = H_i^{-1}B_iH_i$ and, hence, $B' = H^{-1}BH$. Therefore $A' = T'^{-1}H^{-1}BHT' = T'^{-1}H^{-1}T^{-1}ATHT' = C^{-1}AC$, where $C = THT'$ is a symmetric nonsingular matrix. Set $D = C^{-1}A$. Then $D' = A'C'^{-1} = C^{-1}ACC^{-1} = D$. Thus the matrix D is also symmetric and $A = CD$.

1145. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of the matrix A (with account taken of multiplicity), then the eigenvalues of the matrix A_p (also with account taken of multiplicity) are equal to the values $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_p}$ $(1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n)$, that is to say, to all possible products made up of p eigenvalues of A.

Hint. Establish the fact that with a change in the numbering of the combinations of p numbers selected from the n numbers $1, 2, \ldots, n$, the matrix A_p becomes a similar matrix, then use problem 970, pass to the Jordan form A_j, and apply the properties of associated matrices of problem 969.

1146. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are eigenvalues of A and $\mu_1, \ldots, \mu_q$ are eigenvalues of B, then the eigenvalues of $A \times B$ are equal to $\lambda_i\mu_j$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$.

Solution. Suppose the Kronecker product $A \times B$ is determined by the arrangement of $\alpha_1, \alpha_2, \ldots, \alpha_{pq}$ pairs of numbers (i, j) $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$. The transposition of α_i and α_j produces in the matrix $A \times B$ an interchange of the ith and jth rows and the ith and jth columns and, hence, will carry that matrix into a similar matrix (problem 1049). Since any permutation reduces to a series of transpositions, the eigenvalues of all Kronecker products $A \times B$ coincide and one can, for example, consider the right direct product $A \times B$ (problem 964). Let A be equal to $C^{-1}A_jC$ and $B = D^{-1}B_jD$, where A_j and B_j are Jordan matrices. Applying the property (c) of problem 963, we find: $A \times {}^{\cdot}B = C^{-1}A_jC \times {}^{\cdot}D^{-1}B_jD = (C^{-1} \times {}^{\cdot}D^{-1})(A_j \times {}^{\cdot}B_j)(C \times {}^{\cdot}D)$. By property (e) of problem 964 we have $C^{-1} \times {}^{\cdot}D^{-1} = (C \times {}^{\cdot}D)^{-1}$. Hence, the matrices $A \times {}^{\cdot}B$ and $A_j \times {}^{\cdot}B_j$ are similar and their eigenvalues coincide. But $A_j \times {}^{\cdot}B_j$ is a triangular matrix with elements $\lambda_i\mu_j$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$ on the main diagonal, and this completes the proof of our assertion.

1147. *Hint.* Show that $g(A) = h(A)$ if and only if $g(\lambda) - h(\lambda)$ is divisible by $\psi(\lambda)$.

1149. In the given case the minimal polynomial coincides with the characteristic polynomial (up to sign). $r(\lambda)$ is an ordinary Lagrangian interpolation polynomial,

$$f(A) = \sum_{k=1}^{n} f(\lambda_k) \frac{(A-\lambda_1E)\ldots(A-\lambda_{k-1}E)(A-\lambda_{k+1}E)\ldots(A-\lambda_nE)}{(\lambda_k-\lambda_1)\ldots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\ldots(\lambda_k-\lambda_n)},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of matrix A (by hypothesis, distinct).

1150. $r(\lambda)$ is an ordinary Lagrangian interpolation polynomial

$$f(A) = \sum_{k=1}^{s} f(\lambda_k) \frac{(A-\lambda_1E)\ldots(A-\lambda_{k-1}E)(A-\lambda_{k+1}E)\ldots(A-\lambda_sE)}{(\lambda_k-\lambda_1)\ldots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\ldots(\lambda_k-\lambda_s)}.$$

1151. *Solution.* First, we show that if the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ exists, then it is determined by the equalities (1) and (2). Let

$$\frac{r(\lambda)}{\psi(\lambda)}=\sum_{k=1}^{s}\left[\frac{\alpha_{k,1}}{(\lambda-\lambda_k)^{r_k}}+\ldots+\frac{\alpha_{k,r_k}}{(\lambda-\lambda_k)}\right] \tag{3}$$

be an expansion in partial fractions. Multiplying this equality by $\psi(\lambda)$, we obtain (1). To establish (2), let us multiply (3) by $(\lambda-\lambda_k)^{r_k}$ to get

$$\frac{r(\lambda)}{\psi_k(\lambda)}=\alpha_{k,1}+\alpha_{k,2}(\lambda-\lambda_k)+\ldots+\alpha_{k,r_k}(\lambda-\lambda_k)^{r_k-1}$$
$$+(\lambda-\lambda_k)^{r_k}\varphi(\lambda), \tag{4}$$

where $\varphi(\lambda)$ is a rational function that is meaningful when $\lambda=\lambda_k$ together with all its derivatives. Taking the $(j-1)$th derivative of both sides of (4) for $\lambda=\lambda_k$ and using the fact that the values of $r(\lambda)$ and $f(\lambda)$ on the spectrum of matrix A coincide, we thus obtain (2). Secondly, we show that the polynomial $r(\lambda)$ defined by (1) and (2) is a Lagrange-Sylvester interpolation polynomial for the function $f(\lambda)$ on the spectrum of matrix A. From (1) it is evident that the degree of $r(\lambda)$ is lower than that of $\psi(\lambda)$. Furthermore, we set

$$\varphi_k(\lambda)=\alpha_{k,1}+\alpha_{k,2}(\lambda-\lambda_k)+\ldots+\alpha_{k,r_k}(\lambda-\lambda_k)^{r_k-1}.$$

From the equalities (2) it follows that for $\lambda=\lambda_k$ the values of the function $\varphi_k(\lambda)$ and its derivatives of order $j<r_k$ coincide respectively with the values of the function $\frac{f(\lambda)}{\psi_k(\lambda)}$ and of its derivatives of the same order. Therefore, setting $\lambda=\lambda_k$ in $r(\lambda)=\sum_{k=1}^{s}\varphi_k(\lambda)\psi_k(\lambda)$ and the equalities obtained from it by j-fold differentiation $(j<r_k)$, we obtain

$$r^{(j)}(\lambda_k)=f^{(j)}(\lambda_k)\quad(j=0,\ 1,\ \ldots,\ r_k-1;\ k=1,\ 2,\ \ldots,\ s),$$

which is to say the values of $r(\lambda)$ and $f(\lambda)$ coincide on the spectrum of the matrix.

1152. $f(A)=[aE+b(A-\lambda_1E)](A-\lambda_2E)^3+[cE+d(A-\lambda_2E)+$
$+e(A-\lambda_2E)^2](A-\lambda_1E)^2$, where $a=\frac{f(\lambda_1)}{(\lambda_1-\lambda_2)^3}$, $b=-\frac{2}{(\lambda_1-\lambda_2)^4}\times$
$\times f(\lambda_1)+\frac{1}{(\lambda_1-\lambda_2)^3}f'(\lambda_1)$, $c=\frac{f(\lambda_2)}{(\lambda_2-\lambda_1)^2}$, $d=-\frac{2}{(\lambda_2-\lambda_1)^3}f(\lambda_2)+$
$+\frac{1}{(\lambda_2-\lambda_1)^2}f'(\lambda_2)$, $e=\frac{3}{(\lambda_2-\lambda_1)^4}f(\lambda_2)-\frac{2}{(\lambda_2-\lambda_1)^3}f'(\lambda_2)+$
$+\frac{1}{2!}\frac{1}{(\lambda_2-\lambda_1)^2}f''(\lambda_2)$.

1154. *Hint.* Show that the values of the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ for $f(\lambda)$ coincide on the spectrum of matrix A with the values of $f(\lambda)$ on the spectrum of each block of A_k, and apply problem 1147.

1155. $r(\lambda)=f(0)+f'(0)\lambda+\frac{f''(0)}{2!}\lambda^2+\ldots+\frac{f^{(n-1)}(0)}{(n-1)!}\lambda^{n-1}$.

$$f(A)=\begin{pmatrix} f(0) & f'(0) & \frac{f''(0)}{2!} & \ldots & \frac{f^{(n-1)}(0)}{(n-1)!} \\ 0 & f(0) & f'(0) & \ldots & \frac{f^{(n-2)}(0)}{(n-2)!} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & f(0) \end{pmatrix}.$$

$f(A)$ is meaningful for any function $f(\lambda)$ for which the values $f(0)$, $f'(0)$, ..., $f^{(n-1)}(0)$ are defined.

1156. $r(\lambda)=f(\alpha)+f'(\alpha)(\lambda-\alpha)+\frac{f''(\alpha)}{2!}(\lambda-\alpha)^2+\ldots$

$$\ldots+\frac{f^{(n-1)}(\alpha)}{(n-1)!}(\lambda-\alpha)^{n-1}.$$

$$f(A)=\begin{pmatrix} f(\alpha) & f'(\alpha) & \frac{f''(\alpha)}{2!} & \ldots & \frac{f^{(n-1)}(\alpha)}{(n-1)!} \\ 0 & f(\alpha) & f'(\alpha) & \ldots & \frac{f^{(n-2)}(\alpha)}{(n-2)!} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & f(\alpha) \end{pmatrix}.$$

$f(A)$ is meaningful for any function $f(\lambda)$ for which the values $f(\alpha)$, $f'(\alpha)$, ..., $f^{(n-1)}(\alpha)$ exist.

1160. *Hint.* Apply the preceding problem.

1162. $\begin{pmatrix} 3\cdot 2^{100}-2\cdot 3^{100} & 2(3^{100}-2^{100}) \\ -3(3^{100}-2^{100}) & 3^{101}-2^{101} \end{pmatrix}$. 1163. $2^{50}\begin{pmatrix} -24 & 25 \\ -25 & 26 \end{pmatrix}$.

1164. $\pm\frac{1}{4}\begin{pmatrix} 7 & 1 \\ -1 & 9 \end{pmatrix}$. 1165. $\pm\frac{1}{5}\begin{pmatrix} 12 & 2 \\ 3 & 13 \end{pmatrix}$, $\pm\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$; a total of four matrices.

1166. $\begin{pmatrix} 4e-3 & 2-2e \\ 6e-6 & 4-3e \end{pmatrix}$. 1167. $\begin{pmatrix} 2e^2 & -e^2 \\ e^2 & 0 \end{pmatrix}$.

1168. $\begin{pmatrix} 3e-1 & e & -3e+1 \\ 3e & e+3 & -3e-3 \\ 3e-1 & e+1 & -3e \end{pmatrix}=(e-2)A^2+A+E$.

1169. $\begin{pmatrix} 3 & -15 & 6 \\ 1 & -5 & 2 \\ 1 & -5 & 2 \end{pmatrix}$ if we take the real value of the logarithm.

The general solution is of the form

$$\begin{pmatrix} 3+2\pi in & -15 & 6 \\ 1 & -5+2\pi in & 2 \\ 1 & -5 & 2+2\pi in \end{pmatrix},$$

where $i=\sqrt{-1}$ and n is any integer.

1170. $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

1171. *Hint.* Make use of problem 1159.

1172. *Hint.* Make use of problem 1159.

1173. $|e^A| = e^s$, where $s = a_{11} + a_{22} + \ldots + a_{nn}$ is the trace of matrix A.

1174. *Hint.* Use the problem 1161.

1175. $y_1^2+y_2^2-y_3^2$. 1176. $y_1^2-y_2^2-y_3^2$. 1177. $y_1^2-y_2^2$.

1178. $y_1^2-y_2^2-y_3^2-y_4^2$. 1179. $y_1^2+y_2^2-y_3^2$. 1180. $y_1^2+y_2^2-y_3^2$; $x_1=y_1-\frac{1}{2}y_2+\frac{5}{6}y_3$, $x_2=\frac{1}{2}y_2-\frac{1}{6}y_3$, $x_3=\frac{1}{3}y_3$.

1181. $y_1^2+y_2^2-y_3^2$; $x_1=\frac{1}{2}y_1+y_2$, $x_2=y_2+y_3$, $x_3=-y_2+y_3$.

1182. $y_1^2-y_2^2-y_3^2$; $x_1=y_1-y_2-y_3$, $x_2=y_1+y_2-y_3$, $x_3=y_3$.

1183. $y_1^2+y_2^2-y_3^2$; $x_1=\frac{1}{2}\sqrt{2}y_1-\frac{5}{3}\sqrt{3}y_2+\frac{1}{3}\sqrt{3}y_3$, $x_2=-\frac{1}{3}\sqrt{3}y_2+\frac{1}{3}\sqrt{3}y_3$, $x_3=\frac{1}{3}\sqrt{3}y_2+\frac{1}{3}\sqrt{3}y_3$.

1184. $y_1^2+y_2^2-y_3^2$; $x_1=-\frac{3}{4}y_1-\frac{1}{4}y_2+\frac{1}{6}\sqrt{3}y_3$, $x_2=-\frac{1}{2}y_1+\frac{1}{2}y_2$, $x_3=\frac{1}{2}y_1+\frac{1}{2}y_2$.

1185. $y_1^2-y_2^2$; $x_1=y_1-y_2-y_3$, $x_2=y_1+y_2-y_4$, $x_3=y_3$, $x_4=y_4$.

1186. $y_1^2+y_2^2-y_3^2-y_4^2$; $x_1=\frac{1}{3}\sqrt{3}y_1-\frac{1}{15}\sqrt{15}y_2+\frac{2}{85}\sqrt{85}y_3-\frac{1}{629}\sqrt{629}y_4$, $x_2=\frac{1}{5}\sqrt{15}y_2-\frac{6}{85}\sqrt{85}y_3+\frac{3}{629}\sqrt{629}y_4$, $x_3=\frac{1}{17}\sqrt{85}y_3+\frac{6}{629}\sqrt{629}y_4$, $x_4=\frac{1}{37}\sqrt{629}y_4$.

1187. $2y_1^2+10y_2^2+190y_3^2$, $y_1=x_1-\frac{1}{2}x_2+x_3$, $y_2=\frac{1}{2}x_2-\frac{1}{10}x_3$, $y_3=\frac{1}{10}x_3$.

1188. $3y_1^2-30y_2^2+530y_3^2$; $y_1=x_1+\frac{2}{3}x_2-\frac{1}{2}x_3$, $y_2=\frac{1}{3}x_2-\frac{1}{20}x_3$, $y_3=\frac{1}{20}x_3$. 1189. $2y_1^2+6y_2^2-6y_3^2+2y_4^2$; $y_1=\frac{1}{2}x_1-\frac{1}{2}x_2$, $y_2=\frac{1}{2}x_2+\frac{1}{6}x_3$, $y_3=\frac{1}{6}x_3+\frac{1}{2}x_4$, $y_4=\frac{3}{2}x_4$.

1190. $x_1=y_1-3y_2-6y_3$; $x_2=y_2+3y_3$; $x_3=y_3$.

1191. $x_1=2\sqrt{2}y_1+\sqrt{2}y_2+5y_3$; $x_2=\frac{1}{2}\sqrt{2}y_1+y_3$; $x_3=y_3$.

1192. $x_1=y_3$, $x_2=\sqrt{2}y_2+y_3$, $x_3=\sqrt{2}y_1-\frac{3}{2}\sqrt{2}y_2-$

$-\left(3+\frac{3}{2}\sqrt{2}\right)y_3$.

1193. y_1^2; $y_1=a_1x_1+a_2x_2+\ldots+a_nx_n$; $y_2=x_2$; $y_3=x_3$; $\ldots$; $y_{i-1}=$ $=x_{i-1}$; $y_i=x_1$; $y_{i+1}=x_{i+1}$; $\ldots$; $y_n=x_n$, if $a_i\neq 0$.

1194. $y_1^2+\frac{3}{4}y_2^2+\frac{4}{6}y_3^2+\frac{5}{8}y_4^2+\ldots+\frac{n+1}{2n}y_n^2$;

$$y_1=x_1+\frac{1}{2}(x_2+x_3+\ldots+x_n);$$

$$y_2=x_2+\frac{1}{3}(x_3+x_4+\ldots+x_n);$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_n=x_n.$$

1195. $y_1^2-y_2^2-y_3^2-\frac{3}{4}y_4^2-\frac{4}{6}y_5^2-\frac{5}{8}y_6^2-\ldots-\frac{n-1}{2(n-2)}y_n^2$;

$$y_1=\frac{1}{2}(x_1+x_2)+x_3+x_4+\ldots+x_n;$$

$$y_2=\frac{1}{2}(x_1-x_2);$$

$$y_3=x_3+\frac{1}{2}(x_4+x_5+\ldots+x_n);$$

$$y_4=x_4+\frac{1}{3}(x_5+x_6+\ldots+x_n);$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_n=x_n.$$

Hint. Reduce it to the preceding problem.

1196. If n is even: $y_1^2-y_2^2+y_3^2-y_4^2+\ldots+y_{n-1}^2-y_n^2$;

$$y_i=\frac{x_i+x_{i+1}+x_{i+2}}{2}\ (i=1,\ 3,\ 5,\ \ldots,\ n-3);$$

$$y_i=\frac{x_{i-1}-x_i+x_{i+1}}{2}\ (i=2,\ 4,\ 6,\ \ldots,\ n-2);$$

$$y_{n-1}=\frac{x_{n-1}+x_n}{2},\quad y_n=\frac{x_{n-1}-x_n}{2}.$$

If n is odd: $y_1^2-y_2^2+y_3^2-y_4^2+\ldots+y_{n-2}^2-y_{n-1}^2$;

$$y_l=\frac{x_l+x_{l+1}+x_{l+2}}{2} \quad (l=1,\ 3,\ 5,\ \ldots,\ n-2);$$

$$y_l=\frac{x_{l-1}-x_l+x_{l+1}}{2} \quad (i=2,\ 4,\ 6,\ \ldots,\ n-1);$$

$$y_n=x_n.$$

1197. $\frac{n-1}{n}\,y_1^2+\frac{n-2}{n-1}\,y_2^2+\ldots+\frac{2}{3}\,y_{n-2}^2+\frac{1}{2}\,y_{n-1}^2$;

$$y_1=x_1-\frac{x_2+x_3+\ldots+x_n}{n-1};$$

$$y_2=x_2-\frac{x_3+x_4+\ldots+x_n}{n-2};$$

. . . . . .

$$y_{n-1}=x_{n-1}-x_n;$$

$$y_n=x_n.$$

Hint. Represent the form as $f_1=\frac{n-1}{n}\cdot\sum\limits_{i=1}^{n}x_i^2-\frac{2}{n}\sum\limits_{i<j}^{n}x_ix_j$ and apply the method of induction. An alternative approach: perform the transformation $z_1=x_1-s$; $z_2=x_2-s$; $\ldots$; $z_{n-1}=x_{n-1}-s$; $z_n=x_n$; then add the equalities and reduce the form $\sum\limits_{i=1}^{n}(x_i-s)^2$ to $2\left(\sum\limits_{i=1}^{n-1}x_i^2+\sum\limits_{i<j}^{n-1}x_ix_j\right)$. Make use of the answer to problem 1194 to obtain $2y_1^2+\frac{3}{2}\,y_2^2+\frac{4}{3}\,y_3^2+\ldots+\frac{n}{n-1}\,y_{n-1}^2$. In this case, the relationship between the old and new unknowns turns out to be rather involved.

1198. $(n-1)\,y_1^2-y_2^2-y_3^2-\ldots-y_n^2$;

$$y_1=\frac{1}{2}(x_1+x_2+x_3+\ldots+x_n);$$

$$y_2=\frac{1}{2}(-x_1+x_2+x_3+\ldots+x_n);$$

$$y_3=\frac{1}{2}(-x_1-x_2+x_3+\ldots+x_n);$$

. . . . . .

$$y_n=\frac{1}{2}(-x_1-x_2-\ldots-x_{n-1}+x_n).$$

The inverse transformation is of the form: $x_1 = y_1 - y_2$; $x_2 = y_2 - y_3$; $\ldots$; $x_{n-1} = y_{n-1} - y_n$; $x_n = y_1 + y_n$. *Hint.* Apply the transformation

$$\begin{aligned} z_1 &= x_1 + x_2 + \ldots + x_n; \\ z_2 &= x_2 + x_3 + \ldots + x_n; \\ &\ldots\ldots\ldots\ldots \\ z_n &= x_n. \end{aligned}$$

1199. *Hint.* The proof is similar to that of the law of inertia.

1200. *Hint.* Make use of the preceding problem.

1201. The forms f_1 and f_3 are equivalent to each other but are not equivalent to the form f_2.

1202. The forms f_2 and f_3 are equivalent to each other but are not equivalent to form f_1.

1203. In the complex domain $n+1$; in the real domain $\frac{(n+1)(n+2)}{2}$.

1204. The rank is an even number, the signature is zero.

1205. $\left[\frac{n-|s|}{2}\right]+1$, where $[x]$ stands for the largest integer not exceeding x.

1210. *Hint.* To prove (b) consider the form $f_\varepsilon = f + \varepsilon g$, where $\varepsilon > 0$ and g is the sum of the squares of the unknowns. When proving necessity, verify that $f_\varepsilon > 0$ and that for the appropriate principal minors D and D_ε of the forms f and f_ε we have $D = \lim\limits_{\varepsilon \to 0} D_\varepsilon$. When proving sufficiency, expand D_ε in powers of ε, $D_\varepsilon > 0$ and check to see that for arbitrary values of the unknowns we have $f = \lim\limits_{\varepsilon \to 0} f_\varepsilon$.

Examples. The form $f_1 = -x_1^2$ or the nondegenerate form $f_2 = -x_1^2 + 2x_1x_2$ has corner minors that are nonnegative, yet the forms themselves are not nonnegative.

When proving (c) make use of problem 1208.

When proving (d) make use of problem 916 and the reduction of f to normal form by showing that $A = D'BD = (BD)'(BD)$, where D is the transformation matrix and B is the matrix of the form in its normal aspect.

1212. $\lambda > 2$. 1213. $|\lambda| < \sqrt{\frac{5}{3}}$. 1214. $-0.8 < \lambda < 0$.

1215. There are no such values of λ.

1216. There are no such values of λ.

1217. *Hint.* Let $g = f + l^2$, where $l = c_1x_1 + \ldots + c_nx_n$. Change the order of the unknowns so as to reach the case $c_n \neq 0$, perform the transformation $y_i = x_i$ $(i = 1, 2, \ldots, n-1)$, $y_n = \frac{l}{c_n}$ and prove that for the new forms $D_{g_1} = D_{f_1} + c_n^2 D_{n-1}$, where D_{n-1} is a corner minor of order $n-1$ of the form f_1.

1218. *Hint.* Represent the form f as

$$f = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \ldots + \frac{a_{1n}}{a_{11}}x_n\right)^2 + f_1(x_2, \ldots, x_n)$$

and use the preceding problem to show that $D_f = a_{11}D_{f_1} \leqslant a_{11}D_\varphi$

1219. *Hint.* Use the canonical form of the given form.

1220. *Solution.* It is obvious that $(f_1+f_2, g)=(f_1, g)+(f_2, g)$. Reducing both forms to the normal form, we find $f=\sum_{i=1}^{r} y_i^2$, $g=\sum_{j=1}^{s} z_j^2$, where y_i, z_j are linear forms in $x_1, x_2, \ldots, x_n$. By virtue of the noted properties of a composition, we have $(f, g)=\sum_{i=1}^{r}\sum_{j=1}^{s}(y_i^2, z_j^2)$.

Consider one of the summands (y^2, z^2), where $y=\sum_{k=1}^{n} a_k x_k$, $z=\sum_{k=1}^{n} b_k x_k$. Then $y^2=\sum_{k, l=1}^{n} a_k a_l x_k x_l$; $z^2=\sum_{k, l=1}^{n} b_k b_l x_k x_l$; $(y^2, z^2)=\sum_{k, l=1}^{n} a_k a_l b_k b_l x_k x_l=\left(\sum_{k=1}^{n} a_k b_k x_k\right)^2 \geqslant 0$ for arbitrary real values of $x_1, \ldots, x_n$. Whence $(f, g) \geqslant 0$ and this proves (a). Now let $f>0$ and $g>0$. We reduce the form g to the normal form $g=\sum_{i=1}^{n} y_i^2$, where $y_i=\sum_{j=1}^{n} q_{ij}x_j$ $(i=1, \ldots, n)$ and $|Q|=|q_{ij}| \neq 0$. Then $(f, g)=\sum_{i=1}^{n}(f, y_i^2)$. But $y_i^2=\sum_{j, k=1}^{n} q_{ij}q_{ik}x_j x_k$, whence $(f, y_i^2)=\sum_{j, k=1}^{n} a_{jk}q_{ij}q_{ik}x_j x_k=\sum_{j, k=1}^{n} a_{jk}(q_{ij}x_j)(q_{ik}x_k) \geqslant 0$ by virtue of $f>0$. If for a certain i we take the value $x_i \neq 0$, then there is an i such that $q_{ii} \neq 0$ (otherwise we would have $Q=0$ between vertical bars). Hence, since $f>0$, it is also true that $(f, y_i^2)=\sum_{j, k=1}^{n} a_{jk} \times (q_{ij}x_j)(q_{ik}x_k)>0$ and $(f, g)>0$.

1221. *Hint.* When proving (b), consider the forms $f_k=\sum_{i, j=1}^{k} a_{ij}x_i x_j$ $(k=1, 2, \ldots, n)$.

1222. *Hint.* The necessity of the conditions (2) follows from the invariability of the corner minors under triangular transformations (see the preceding problem).

The equalities (3) are proved in the same way. Sufficiency can be proved by induction on the number of unknowns n.

1224. $f_1=-2y_1^2+\frac{2}{3}y_2^2$; $g_1=y_1^2+y_2^2$; $x_1=y_1+\frac{1}{3}\sqrt{3}y_2$; $x_2=\frac{1}{2}y_1-\frac{1}{6}\sqrt{3}y_2$.

1225. $f_1 = y_1^2 + y_2^2$; $g_1 = 4y_1^2 - 2y_2^2$; $x_1 = -2\sqrt{2}y_1 + 3\sqrt{2}y_2$; $x_2 = \frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_2$.

1226. $f_1 = 9y_1^2 + 9y_2^2 - 9y_3^2$; $g_1 = y_1^2 + y_2^2 + y_3^2$; $x_1 = \sqrt{2}y_2$; $x_2 = \frac{1}{6}y_1 - \frac{1}{3}\sqrt{2}y_3$; $x_3 = \frac{2}{3}y_1 - \frac{1}{2}\sqrt{2}y_2 + \frac{1}{6}\sqrt{2}y_3$.

1227. $f_1 = y_1^2 + y_2^2 + y_3^2 - 3y_4^2$; $g_1 = y_1^2 + y_2^2 + y_3^2 + y_4^2$; $x_1 = y_1 + y_2 + y_3 + y_4$; $x_2 = y_2 - y_4$; $x_3 = \frac{1}{2}y_1 - \frac{1}{2}y_2 + \frac{1}{2}y_3 - \frac{1}{2}y_4$; $x_4 = \frac{1}{2}y_1 - \frac{1}{2}y_2 - \frac{1}{2}y_3 + \frac{1}{2}y_4$.

1228. $f_1 = y_1^2 + 2y_2^2 + 2y_3^2 - 7y_4^2$; $g_1 = y_1^2 + y_2^2 + y_3^2 + y_4^2$; $x_1 = \frac{2}{3}y_2 + \frac{2}{3}y_3 + \frac{1}{3}y_4$; $x_2 = \frac{2}{3}y_2 - \frac{4}{3}y_3 + \frac{4}{3}y_4$; $x_3 = y_3 - 2y_4$; $x_4 = y_1$.

1229. $f_1 = y_1^2 + 2y_2^2 - 3y_3^2$; $g_1 = y_1^2 + y_2^2 + y_3^2$; $x_1 = y_1 - y_2$; $x_2 = -y_2 + y_3$; $x_3 = -3y_2 + 2y_3$.

1230. *Hint.* Show that the roots of the λ-equation of a pair of forms do not change under any nonsingular linear transformation of the unknowns.

1231. Impossible since the roots of the λ-equation are $1 \pm \frac{1}{2}i$.

1232. Impossible since the roots of the λ-equation are $\pm\frac{1}{2}i\sqrt{5}$.

1233. For a suitable numbering we have $\lambda_i\mu_i = 1$ ($i = 1, 2, \ldots, n$).

1234. $3y_1^2 - y_2^2 - 5y_3^2$. 1235. $5y_1^2 - y_2^2 - 2y_3^2$. 1237. They are equivalent.

1238. They are equivalent.

1239. $x_1 = -12y_1 - 17y_2$; $x_2 = 5y_1 + 7y_2$. 1240. $x_1 = y_1 + 2y_2$; $x_2 = 3y_1 + 2y_2$.

1242. *Hint.* Show that the characteristic polynomial $|A - \lambda E|$ does not change under an orthogonal transformation of the form f.

1243. $4y_1^2 + 4y_2^2 - 2y_3^2$. 1244. $6y_1^2 + 6y_2^2 + 9y_3^2$.

1245. $y_1^2 + \sqrt{3}y_2^2 - \sqrt{3}y_3^2$. 1246. $3y_1^2 + (1+\sqrt{17})y_2^2 + (1-\sqrt{17})y_3^2$.

1247. $\sum_{k=1}^{n} \cos\frac{\pi k}{n+1} y_k^2$. *Hint.* Consider a doubled form and solve in the same manner as problem 1084.

1248. $3y_1^2 + 6y_2^2 + 9y_3^2$, $x_1 = \frac{2}{3}y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_3$; $x_2 = \frac{2}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3$; $x_3 = -\frac{1}{3}y_1 + \frac{2}{3}y_2 + \frac{2}{3}y_3$.

1249. $9y_1^2 + 18y_2^2 - 9y_3^2$; $x_1 = \frac{2}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3$; $x_2 = -\frac{1}{3}y_1 + \frac{2}{3}y_2 + \frac{2}{3}y_3$; $x_3 = \frac{2}{3}y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_3$.

1250. $3y_1^2+6y_2^2-2y_3^2$; $x_1=\frac{1}{\sqrt{3}}y_1+\frac{1}{\sqrt{6}}y_2+\frac{1}{\sqrt{2}}y_3$; $x_2=-\frac{1}{\sqrt{3}}y_1-$ $-\frac{1}{\sqrt{6}}y_2+\frac{1}{\sqrt{2}}y_3$; $x_3=\frac{1}{\sqrt{3}}y_1-\frac{2}{\sqrt{6}}y_2$.

1251. $5y_1^2-y_2^2-y_3^2$; $x_1=\frac{1}{\sqrt{3}}y_1+\frac{1}{\sqrt{6}}y_2+\frac{1}{\sqrt{2}}y_3$; $x_2=\frac{1}{\sqrt{3}}y_1+$ $+\frac{1}{\sqrt{6}}y_2-\frac{1}{\sqrt{2}}y_3$; $x_3=\frac{1}{\sqrt{3}}y_1-\frac{2}{\sqrt{6}}y_2$.

1252. $9y_1^2+18y_2^2+18y_3^2$; $x_1=\frac{1}{3}y_1-\frac{2}{3}y_2+\frac{2}{3}y_3$; $x_2=\frac{2}{3}y_1-$ $-\frac{1}{3}y_2-\frac{2}{3}y_3$; $x_3=\frac{2}{3}y_1+\frac{2}{3}y_2+\frac{1}{3}y_3$.

1253. $3y_1^2-6y_2^2$; $x_1=\frac{2}{3}y_1+\frac{1}{6}\sqrt{2}y_2+\frac{1}{2}\sqrt{2}y_3$; $x_2=\frac{1}{3}y_1-$ $-\frac{2}{3}\sqrt{2}y_2$; $x_3=\frac{2}{3}y_1+\frac{1}{6}\sqrt{2}y_2-\frac{1}{2}\sqrt{2}y_3$.

1254. $9y_1^2+9y_2^2-9y_3^2$; $x_1=\frac{2}{3}y_1+\frac{1}{2}\sqrt{2}y_2+\frac{1}{6}\sqrt{2}y_3$; $x_2=$ $=\frac{1}{3}y_1-\frac{2}{3}\sqrt{2}y_3$; $x_3=\frac{2}{3}y_1-\frac{1}{2}\sqrt{2}y_2+\frac{1}{6}\sqrt{2}y_3$.

1255. $2y_1^2+4y_2^2-2y_3^2-4y_4^2$; $x_1=\frac{1}{2}(y_1+y_2+y_3+y_4)$; $x_2=$ $=\frac{1}{2}(-y_1+y_2+y_3-y_4)$; $x_3=\frac{1}{2}(-y_1-y_2+y_3+y_4)$; $x_4=$ $=\frac{1}{2}(y_1-y_2+y_3-y_4)$.

1256. $4y_1^2+8y_2^2+12y_3^2-4y_4^2$; $x_1=\frac{1}{2}(y_1+y_2+y_3+y_4)$; $x_2=$ $=\frac{1}{2}(y_1-y_2-y_3+y_4)$; $x_3=\frac{1}{2}(y_1+y_2-y_3-y_4)$; $x_4=\frac{1}{2}(y_1-$ $-y_2+y_3-y_4)$.

1257. $5y_1^2-5y_2^2+5y_3^2$; $x_1=\frac{1}{5}\sqrt{5}(2y_1+y_2)$; $x_2=\frac{1}{5}\sqrt{5}(y_1-2y_2)$; $x_3=\frac{1}{5}\sqrt{5}(2y_3+y_4)$; $x_4=\frac{1}{5}\sqrt{5}(-y_3+2y_4)$.

1258. $2y_1^2-4y_2^2$; $x_1=\frac{1}{2}\sqrt{2}(y_1+y_3)$; $x_2=\frac{1}{2}\sqrt{2}(y_1-y_3)$; $x_3=$ $=\frac{1}{2}\sqrt{2}(y_2+y_4)$; $x_4=\frac{1}{2}\sqrt{2}(y_2-y_4)$.

1259. $9y_1^2+9y_2^2+9y_3^2$; $x_1=y_1$; $x_2=\frac{1}{3}(y_2+2y_3+2y_4)$; $x_3=$ $=\frac{1}{3}(2y_2+y_3-2y_4)$; $x_4=\frac{1}{3}(2y_2-2y_3+y_4)$.

1260. $5y_1^2+5y_2^2+5y_3^2-8y_4^2$; $x_1=\frac{1}{5}\sqrt{5}\,(2y_1+y_5)$; $x_2=\frac{1}{5}\sqrt{5}\times$ $\times(-y_1+2y_5)$; $x_3=y_3$; $x_4=\frac{1}{13}\sqrt{13}\,(2y_2+3y_4)$; $x_5=\frac{1}{13}\sqrt{13}\times$ $\times(3y_2-2y_4)$.

1261. $4y_1^2+4y_2^2+4y_3^2-6y_4^2-6y_5^2$; $x_1=y_1$; $x_2=\frac{1}{5}\sqrt{5}\,(y_3+2y_4)$; $x_3=\frac{1}{5}\sqrt{5}\,(-2y_3+y_4)$; $x_4=\frac{1}{10}\sqrt{10}\,(y_3+3y_5)$; $x_5=\frac{1}{10}\sqrt{10}\times$ $\times(3y_3-y_5)$.

1262. $5y_1^2-5y_2^2+5y_3^2-5y_4^2+5y_5^2$; $x_1=\frac{1}{5}\sqrt{5}\,(2y_1+y_2)$; $x_2=$ $=\frac{1}{5}\sqrt{5}\,(y_1-2y_2)$; $x_3=\frac{1}{10}\sqrt{10}\,(3y_3+y_4)$; $x_4=\frac{1}{10}\sqrt{10}\,(-y_3+$ $+3y_4)$; $x_5=\frac{1}{5}\sqrt{5}\,(2y_5+y_6)$; $x_6=\frac{1}{5}\sqrt{5}\,(y_5-2y_6)$.

1263. $\frac{n+1}{2}y_1^2+\frac{1}{2}y_2^2+\ldots+\frac{1}{2}y_n^2$; $y_1=\frac{1}{\sqrt{n}}(x_1+x_2+\ldots+x_n)$; $y_i=\frac{1}{\sqrt{i(i-1)}}[x_1+x_2+\ldots+x_{i-1}-(i-1)x_i]$ $(i=2,\ 3,\ \ldots,\ n)$.

1264. $\frac{n-1}{2}y_1^2-\frac{1}{2}y_2^2-\frac{1}{2}y_3^2-\ldots-\frac{1}{2}y_n^2$; the same transformation may be taken as in the preceding problem.

1265. *Hint.* Show that under an orthogonal transformation of a quadratic form the characteristic polynomial of its matrix remains unchanged.

1266. The forms f and h are orthogonally equivalent to each other but they are not orthogonally equivalent to the form g.

1267. The forms g and h are orthogonally equivalent to each other but they are not orthogonally equivalent to the form f.

1269.
$$Q=\begin{pmatrix}\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\end{pmatrix};\qquad B=\begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 7\end{pmatrix}.$$

1270.
$$Q=\begin{pmatrix}\frac{2}{5}\sqrt{5} & 0 & \frac{1}{5}\sqrt{5}\\ \frac{2}{15}\sqrt{5} & -\frac{1}{3}\sqrt{5} & -\frac{4}{15}\sqrt{5}\\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\end{pmatrix};\qquad B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 10\end{pmatrix}.$$

1271. *Hint.* Use the hint in problem 1074 to show that the eigenvalues of matrix $A-\lambda_0 E$ are obtained by subtracting λ_0 from the eigenvalues of matrix A, and then apply problem 1242.

1272. *Hint.* Use the preceding problem.

1274. A matrix with positive corner minors is orthogonal if and only if it is a unit matrix.

1275. *Solution.* The quadratic form f with matrix $A'A$ is positive definite (problem 1207); this means that a triangular transformation can reduce it to canonical form with positive coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ (problem 1222). If C is the matrix of that transformation, D a diagonal matrix with elements $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$ on the diagonal (all values of the roots are taken positive), and $B = DC$, then $A'A = C'D^2C = B'B$, where the matrix B satisfies the requirements of the problem. Set $Q = AB^{-1}$. Then $Q'Q = (AB^{-1})' \cdot (AB^{-1}) = (B')^{-1} \times (A'A) \cdot B^{-1} = (B')^{-1} \cdot B'B \cdot B^{-1} = E$, that is, the matrix Q is orthogonal and $A = QB$; and if, besides, $A = Q_1B_1$, then the matrix $Q^{-1} \cdot Q_1 = BB_1^{-1}$ is orthogonal and triangular with positive elements on the diagonal. This means that it is a unit matrix, whence $Q = Q_1$ and $B = B_1$.

1276. *Solution.* We prove assertion (a) for the representation $A = QB$ of the required form. The matrix $A'A$ is symmetric, and the quadratic form of the matrix is positive definite (problem 1207). Therefore there exists an orthogonal matrix P such that $A'A = P'CP$, where C is a diagonal matrix with positive elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal. Let D be a diagonal matrix with the elements $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$ on the diagonal (we take positive values of the root). Set $B = P'DP = P^{-1}DP$. From this it follows that B is a symmetric matrix with positive eigenvalues. Hence the quadratic form with matrix B is positive definite and the corner minors of it are positive. Furthermore, $A'A = P^{-1}CP = P^{-1}D^2P = P^{-1}DP \cdot P^{-1}DP = B^2$. Set $Q = AB^{-1}$. Then $A + QB$ and $Q'Q = (AB^{-1})' \cdot (AB^{-1}) = B'^{-1}(A'A) \cdot B^{-1} = B^{-1}B^2 \cdot B^{-1} = E$, which means the matrix Q is orthogonal.

Suppose we have two representations of the required type: $A = Q_1B_1 = Q_2B_2$. Then $A'A = B_1^2 = B_2^2$. Let us denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$; $\mu_1, \mu_2, \ldots, \mu_n$; $\nu_1, \nu_2, \ldots, \nu_n$ the eigenvalues, respectively, of $A'A$, B_1, B_2 arranged in nonincreasing order. They are all positive and $\mu_i^2 = \lambda_i = \nu_i^2$ $(i = 1, 2, \ldots, n)$ (problem 1077). Hence, $\mu_i = \nu_i$ $(i = 1, 2, \ldots, n)$. Let C and D be diagonal matrices with elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ on the diagonal. There are orthogonal matrices U and V such that $B_1 = U'DU$, $B_2 = V'DV$. Hence, $B_1^2 = U'CU$, $B_2^2 = V'CV$, whence $U'CU = V'CV$, $CUV = UV'C$. The matrix $W = (w_{ij})_1^n = UV'$ commutes with C. Let us now show that it commutes with D. If $\lambda_i \neq \lambda_j$, then, by computing the element of the matrix $CW = WC$ in the ith row and the jth column, we will find that $w_{ij} = 0$. Thus, if we represent the matrix C in the form of a block-diagonal matrix with diagonal blocks $C_1, C_2, \ldots, C_k$ so that in each block the diagonal elements are the same, and are distinct in different blocks, then the matrix W is a block-diagonal matrix with diagonal blocks $W_1, W_2, \ldots, W_k$ of the same orders as $C_1, C_2, \ldots, C_k$. By the construction of matrix D, it too will be a block-diagonal matrix with the diagonal blocks $D_1, D_2, \ldots, D_k$ of the same orders and with identical diagonal elements in each block. Since $D_iW_i = W_iD_i$ $(i = 1, 2, \ldots, k)$, it follows that $DW = WD$, whence $DUV' = UV'D$; $U'DU = V'DV$, that is, $B_1 = B_2$.

The proof is shorter if one uses functions of matrices. If $A = QB$ is the desired representation, then $A'A = B^2$, $B = \sqrt{A'A}$, and the eigenvalues of B are positive. Thus, B is a value of the function $\sqrt{\lambda}$ (where we take the principal value of the root) for $\lambda = A'A$. Since the eigenvalues of the matrix $A'A$ are positive, this value is meaningful, definitely unique and, as a polynomial of the symmetric matrix $A'A$, it is a symmetric matrix (problems 1148, 1151). Putting $Q = AB^{-1}$, we see (as we did above) that Q is orthogonal.

The representation $A = B_2Q_2$ is obtained in similar fashion with the aid of the matrix AA'. The assertion (b) is proved in the same way as (a) with the positive definite forms replaced by Hermitian positive definite forms. Assertion (c) follows from the uniqueness of the representations given in (a) and (b). It can also be demonstrated by reducing matrix B to diagonal form in the case (1) and matrix A in the case (2) via an orthogonal (unitary) matrix (see problem 1595). Then from assertion (1) there readily follows the uniqueness of the representations indicated in (a) and (b).

Chapter IV. Vector Spaces and Their Linear Transformations

1277. (1, 2, 3). **1278.** (1, 1, 1). **1279.** (0, 2, 1, 2).

1280. $x_1 = -27x_1' - 71x_2' - 41x_3'$; $x_2 = 9x_1' + 20x_2' + 9x_3'$; $x_3 = 4x_1' + 12x_2' + 8x_3'$.

1281. $x_1 = 2x_1' + x_3' - x_4'$; $x_2 = -3x_1' + x_2' - 2x_3' + x_4'$; $x_3 = x_1' - 2x_2' + 2x_3' - x_4'$; $x_4 = x_1' - x_2' + x_3' - x_4'$.

1282. (*a*) $a_0, a_1, a_2, \ldots, a_n$; (*b*) $f(\alpha), f'(\alpha), \frac{f''(\alpha)}{2!}, \ldots, \frac{f^{(n)}(\alpha)}{n!}$.

1283.
$$\begin{pmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 & \ldots & (-1)^n \alpha^n \\ 0 & 1 & -2\alpha & 3\alpha^2 & \ldots & (-1)^{n-1} n\alpha^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

In this matrix, the $(k+1)$th column involves the numbers $(-\alpha)^k$, $C_k^{k-1}(-\alpha)^{k-1}$, $C_k^{k-2}(-\alpha)^{k-2}$, ..., $C_k^1(-\alpha)$, 1, 0, 0, 0, ..., 0.

1284. (a) Two rows are interchanged; (b) two columns are interchanged; (c) there will be a symmetric reflection of the matrix about its centre.

1285. No. **1286.** No.

1287. Yes, if the given straight line passes through the origin, no if it does not.

1288. Yes. **1289.** No. **1290.** No. **1291.** Yes.

1292. No. **1293.** Yes.

1294. The entire space; vectors lying in any plane that passes through the coordinate origin; vectors on any straight line that passes through the origin; and the origin itself, that is, the single zero vector.

1295. No, it is not.

1297. The basis is formed, for example, by the vectors (1, 0, 0, ..., 0, 1), (0, 1, 0, ..., 0, 0), (0, 0, 1, ..., 0, 0), ..., (0, 0, 0, ..., 1, 0). The dimension is equal to $n - 1$.

1298. The basis is formed by the following vectors: if k is the number of a basis vector, then its coordinate numbered $2k-1$ is equal to 1, and the other coordinates are equal to zero, $k = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$, where $[x]$ denotes the largest integer not exceeding x. The dimension is equal to $\left[\frac{n+1}{2}\right]$.

1299. The basis is formed by vectors that are designated as basis vectors in the answer to the preceding problem, and another vector is adjoined whose coordinates with even numbers are equal to unity, while those with odd numbers are equal to zero. The dimension is $1 + \left[\frac{n+1}{2}\right]$.

1300. The basis is formed by the vectors $(1, 0, 1, 0, 1, 0, \ldots)$ and $(0, 1, 0, 1, 0, 1, \ldots)$. The dimension is equal to 2.

1301. The basis is formed, for example, by the matrices E_{ij} $(i, j = 1, 2, \ldots, n)$, where E_{ij} is a matrix whose element in the ith row and the jth column is equal to unity; all other elements are zero. The dimension is equal to n^2.

1302. The basis is formed, for example, by the polynomials $1, x, x^2, \ldots, x^n$. The dimension is equal to $n+1$.

1303. The basis is formed, for example, by the matrices F_{ij} $(i \leqslant j;\ i, j = 1, 2, \ldots, n)$, where F_{ij} is a matrix whose elements $f_{ij} = f_{ji} = 1$ and all other elements are zero. The dimension is equal to $\frac{n(n+1)}{2}$.

1304. The basis is formed, for example, by the matrices G_{ij} $(i < j;\ i, j = 1, 2, \ldots n)$, where G_{ij} is a matrix whose elements $g_{ij} = 1$, $g_{ji} = -1$ and all other elements are zero. The dimension is equal to $\frac{n(n-1)}{2}$.

1308. The basis is formed, for example, by the vectors $(1, 0, 0, \ldots, 0, -1)$, $(0, 1, 0, \ldots, 0, -1)$, $\ldots$, $(0, 0, 0, \ldots, 1, -1)$. The dimension is equal to $n-1$.

1310. The dimension is equal to 3. The basis is formed, for example, by the vectors a_1, a_2, a_4.

1311. The dimension is equal to 3. The basis is formed, for example, by the vectors a_1, a_2, a_5.

1312. For example, $x_1 - x_3 - x_4 = 0$, $x_2 + x_3 - x_4 = 0$.

1313. For example, $x_1 - x_2 - 2x_3 = 0$, $x_1 - x_2 + 2x_4 = 0$, $2x_1 + x_2 - x_5 = 0$.

1317. $s = 3$, $d = 1$. **1318.** $s = 3$, $d = 2$.

1319. *Solution.* Rule (1) is readily proved. Let us prove Rule (2). Since the numbers $x_{i1}, \ldots, x_{ik}, y_{i1}, \ldots, y_{il}$ satisfy the equality (1), then $c_i = \sum_{j=1}^{l} y_{ij}b_j = \sum_{j=1}^{k} x_{ij}a_j$. For this reason, the vectors c_i $(i = 1, 2, \ldots, d)$ belong both to L_1 and to L_2 and, hence, to their intersection D. Let x be any vector in D. It can be expressed both in terms of the basis of L_1 and in terms of the basis of L_2. Thus, $x = \alpha_1 a_1 + \ldots + \alpha_k a_k = \beta_1 b_1 + \ldots + \beta_l b_l$. This means that the row of numbers $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$ is a solution of the system

of equations (1) and therefore can be linearly expressed in terms of the fundamental system of solutions (2). Let $\gamma_1, \ldots, \gamma_d$ be the coefficients of that expression. Then $\beta_j = \sum_{i=1}^{d} \gamma_i y_{ij}$ $(j = 1, 2, \ldots, l)$, whence

$$x = \sum_{j=1}^{l} \beta_j b_j = \sum_{j=1}^{l} \left(\sum_{i=1}^{d} \gamma_i y_{ij} \right) b_j = \sum_{i=1}^{d} \gamma_i \left(\sum_{j=1}^{l} y_{ij} b_j \right) = \sum_{i=1}^{d} \gamma_i c_i.$$

Thus, any vector $x \in D$ can be linearly expressed in terms of the vectors (4). Finally, the system (4) is linearly independent, since the matrix of the coordinates of the vectors in the basis $b_1, \ldots, b_l$ contains the nonzero minor (3) of order d.

1320. The basis of the sum is formed, for example, by the vectors a_1, a_2, b_1. The basis of the intersection consists of the single vector $c = 2a_1 + a_2 = b_1 + b_2 = (3, 5, 1)$.

1321. The basis of the sum is formed, for example, by the vectors a_1, a_2, a_3, b_2. The basis of the intersection, for example, by $b_1 = -2a_1 + a_2 + a_3$, $b_2 = 5a_1 - a_2 - 2a_3$.

1322. The basis of the sum consists, for example, of the vectors a_1, a_2, a_3, b_1. The basis of the intersection, for example, consists of the vectors $c_1 = a_1 + a_2 + a_3 = b_1 + b_2 = (1, 2, 2, 1)$, $c_2 = 2a_1 + 2a_3 = b_1 + b_3 = (2, 2, 2, 2)$.

1328. The projection of the vector e_i on L_1 parallel to L_2 has $\frac{n-1}{n}$ as the ith coordinate, all other coordinates being $\frac{-1}{n}$; the projection on L_2 parallel to L_1 has all coordinates equal to $\frac{1}{n}$.

1329.

$$A_1 = \begin{pmatrix} 1 & \frac{1}{2} & \ldots & \frac{1}{2} \\ \frac{1}{2} & 1 & \ldots & \frac{1}{2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{2} & \frac{1}{2} & \ldots & 1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & \frac{1}{2} & \ldots & \frac{1}{2} \\ -\frac{1}{2} & 0 & \ldots & \frac{1}{2} \\ \ldots & \ldots & \ldots & \ldots \\ -\frac{1}{2} & -\frac{1}{2} & \ldots & 0 \end{pmatrix}.$$

1334. *Hint.* Consider the parametric equations of the straight lines $x = a_0 + a_1 t$, $x = b_0 + b_1 t$, where a_0, a_1, b_0, b_1 are given vectors.

1335. The vectors $a_0 - b_0$, a_1, b_1 must be linearly dependent.

1336. The desired conditions are that the vectors a_1 and b_1 are linearly independent and the vector $a_0 - b_0$ can be expressed linearly in terms of a_1 and b_1. If $a_0 - b_0 = t_1 a_1 + t_2 b_1$, then the point of intersection is given by the vector $a_0 - t_1 a_1 = b_0 + t_2 b_1$.

1337. $(-2, -5, -1, 1, -1)$. **1338.** $(0, 1, -1, -2, -3)$.

1339. The necessary and sufficient conditions are that the quadruplet of vectors $a_0 - c$, $b_0 - c$, a_1, b_1 is linearly dependent, and each of the two triplets $a_0 - c$, a_1, b_1 and $b_0 - c$, a_1, b_1 is linearly independent. The desired straight line has the equation $x = c + dt$, where $d = \lambda_1 (a_0 - c) + \lambda_2 a_1 = \lambda_3 (b_0 - c) + \lambda_4 b_1$ and the coefficients λ_1 and λ_3 are nonzero.

The points of intersection of this straight line and the given straight lines are of the form

$$a_0+\frac{\lambda_2}{\lambda_1}a_1=c+\frac{1}{\lambda_1}d \quad \text{and} \quad b_0+\frac{\lambda_4}{\lambda_3}b_1=c+\frac{1}{\lambda_3}d.$$

1340. $x = c + dt$, where $d = (6, 7, -8, -11)$; $M_1(2, 2, -3, -4)$, $M_2(-4, -5, 5, 7)$.

1341. $x = c + dt$, where $d = (1, 1, 0, 3)$; $M_1(2, 3, 2, 1)$, $M_2(1, 2, 2, -2)$.

1344. If two planes of three-dimensional space have a common point, they have a common straight line. If a plane and a three-dimensional linear manifold of four-dimensional space have a common point, then they have a common straight line. If two three-dimensional linear manifolds of four-dimensional space have a common point, they have a plane in common.

1345. For convenience in classifying different cases we introduce two matrices: A is a matrix in which the columns contain the coordinates of the vectors a_1, a_2, b_1, b_2, and B is a matrix obtained from A by adjoining a column of coordinates of the vector $a_0 - b_0$. Let the rank of A be r_1 and the rank of B be r_2. One of six cases is possible:

(1) $r_1 = 4$, $r_2 = 5$. The planes do not lie in a single four-dimensional manifold (the planes are absolutely skew).

(2) $r_1 = r_2 = 4$. The planes have a common point and, hence, lie in one four-dimensional manifold, but do not lie in one three-dimensional manifold (the planes intersect absolutely).

(3) $r_1 = 3$, $r_2 = 4$. The planes do not have any common points, they lie in one four-dimensional manifold, but they do not lie in one three-dimensional manifold (the planes are skew in parallel to a straight line, namely: they are both parallel to a straight line with the equation $a_1t_1 + a_2t_2 = b_1t_3 + b_2t_4$).

(4) $r_1 = r_2 = 3$. The planes lie in three-dimensional space and intersect along a straight line.

(5) $r_1 = 2$, $r_2 = 3$. The planes do not have any points in common, but they lie in one three-dimensional space (the planes are parallel).

(6) $r_1 = r_2 = 2$. The planes are coincident, $r_2 \geqslant r_1 \geqslant 2$, since both pairs of vectors a_1, a_2 and b_1, b_2 are linearly independent.

1347. *Hint.* Carry out the proof by induction on the number k.

1348. An octahedron with vertices at the points

$$(1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1),$$
$$(-1, -1, 1, 1), (-1, 1, -1, 1), (-1, 1, 1, -1).$$

Hint. Note, in determining the coordinates of the vertices, that the vertices of the desired section must be the points of intersection of the cutting subspace with the edges of the cube, and that three coordinates along each edge of the cube are equal to ± 1, while the fourth one varies from $+1$ to -1.

1349. A tetrahedron with vertices at the points $\left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$, $\left(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$, $\left(-\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}\right)$, $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}\right)$. *Hint.* Find the projections of the vertices.

1350. *Hint.* Take the given end of the diagonal for the origin, take the edges issuing from it for the coordinate axes and show that the parallel linear manifolds under consideration are determined by the equations

$$x_1 + x_2 + \ldots + x_n = k \ (k = 0, 1, 2, \ldots, n)$$

while the point of intersection of the diagonal with the kth of these manifolds have all coordinates equal to one and the same number $\frac{k}{n}$ $(k = 0, 1, 2, \ldots, n)$.

1352. The bilinear form g must be symmetric, that is, $a_{ij} = a_{ji}$ $(i, j = 1, 2, \ldots, n)$ and the corresponding quadratic form $f = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ must be positive definite; $(e_ie_j) = a_{ij}$ $(i, j = 1, 2, \ldots, n)$.

1357. They may be completed by adjoining the vectors (2, 2, 1, 0), (5, −2, −6, −1).

1358. They may be completed by adjoining the vectors (1, −2, 1, 0), (25, 4, −17, −6).

1359. One of the vectors $\pm\left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$.

1360. For example, $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2} -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

1361. (1, 2, 2, −1), (2, 3, −3, 2), (2, −1, −1, −2).

1362. (1, 1, −1, −2), (2, 5, 1, 3).

1363. (2, 1, 3, −1), (3, 2, −3, −1), (1, 5, 1, 10).

1366. For example, $b_1 = (2, -2, -1, 0)$, $b_2 = (1, 1, 0, -1)$.

1367. For example, $6x_1 - 9x_2 - x_3 = 0$, $x_2 + x_4 = 0$.

1369. Let $a_1, a_2, \ldots, a_k$ be a basis of L. We seek y in the form $y = \sum_{j=1}^{k} c_ja_j$. Forming the scalar product of this equation by a_i and noting that $(a_i, y) = (a_i, x)$, we obtain the system of equations $\sum_{j=1}^{k} (a_i, a_j)\, c_j = (a_i, x)$ $(i = 1, 2, \ldots, k)$, which, by the meaning of $c_1, c_2, \ldots, c_k$, must have a unique solution. Finding y, we set $z = x - y$.

1370. $y = 3a_1 - 2a_2 = (1, -1, -1, 5)$; $z = (3, 0, -2, -1)$.

1371. $y = 2a_1 - a_2 = (3, 1, -1, -2)$; $z = (2, 1, -1, 4)$.

1372. $y = (5, -5, -2, -1)$; $z = (2, 1, 1, 3)$.

1373. *Hint.* Derive the relation

$$|x - u|^2 = |(x - x_0) - y|^2 + |y - (u - x_0)|^2,$$

where y is the orthogonal projection of $x - x_0$ on L.

1374. (a) 5, (b) 2.

*1375. *Hint.* Set $x - x_0 = y + z$, where $y \in L$, $z \in L^*$. From problem 1373, $d^2 = (z, z)$. Let $y = \lambda_1a_1 + \lambda_2a_2 + \ldots + \lambda_ka_k$. Subtract from the last column of the determinant $G(a_1, a_2, \ldots, a_k,$

$x - x_0$) all preceding columns multiplied respectively by $\lambda_1, \lambda_2, \ldots$ $\ldots, \lambda_k$ and show that the result is zero at the position $(a_i, x - x_0)$ and $(x - x_0, z) = (z, z)$ at the position $(x - x_0, x - x_0)$.

1376. *Hint.* Let $u_1 \in P_1$, $u_2 \in P_2$, y be an orthogonal projection of $x_1 - x_2$ on L. Derive the equation

$$| u_1 - u_2 |^2 = | (x_1 - x_2) - y |^2 + | y + (u_1 - x_1) - (u_2 - x_2) |^2.$$

1377. 3.

1378. *Solution.* Take one of the vertices of the first face as the coordinate origin. Let the other vertices of the first face be given by the vectors $x_1, x_2, \ldots, x_k$, and the vertices of the second face by the vectors $x_{k+1}, \ldots, x_n$. We seek the distance between the linear manifolds determined by the vertices of these faces (problem 1346). These manifolds are

$$x_1 t_1 + \ldots + x_k t_k \quad \text{and} \quad (x_{k+1} - x_n) t_{k+1} + \ldots + (x_{n-1} - x_n) t_{n-1} + x_n.$$

The desired distance is equal to the length of the orthogonal component z of the vector x_n with respect to the subspace L spanned by the vectors $x_1, \ldots, x_k, x_{k+1} - x_n, \ldots, x_{n-1} - x_n$. Multiplying the equation

$$x_n = x_1 t_1 + \ldots + x_k t_k + (x_{k+1} - x_n) t_{k+1} + \ldots + (x_{n-1} - x_n) t_{n-1} + z$$

by the vectors $x_1, \ldots, x_k, x_{k+1} - x_n, \ldots, x_{n-1} - x_n$, we get the following equations:

$$\begin{aligned}
t_1 + \frac{1}{2} t_2 + \ldots + \frac{1}{2} t_k &= \frac{1}{2}; \\
\frac{1}{2} t_1 + t_2 + \ldots + \frac{1}{2} t_k &= \frac{1}{2}; \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\frac{1}{2} t_1 + \frac{1}{2} t_2 + \ldots + t_k &= \frac{1}{2}; \\
t_{k+1} + \frac{1}{2} t_{k+2} + \ldots + \frac{1}{2} t_{n-1} &= -\frac{1}{2}; \\
\frac{1}{2} t_{k+1} + t_{k+2} + \ldots + \frac{1}{2} t_{n-1} &= -\frac{1}{2}; \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\frac{1}{2} t_{k+1} + \frac{1}{2} t_{k+2} + \ldots + t_{n-1} &= -\frac{1}{2},
\end{aligned}$$

whence, adding the first k and the last $n - k - 1$ equations, we readily find $t_1 = \ldots = t_k = \frac{1}{k+1}$; $t_{k+1} = \ldots = t_{n-1} = -\frac{1}{n-k}$. Therefore, $z = \frac{x_{k+1} + \ldots + x_n}{n-k} - \frac{x_1 + \ldots + x_k}{k+1}$, whence it is evident that z joins the centres of the given faces.

Since the distance between manifolds defined by the vertices of the given faces is equal, by what has been proved, to the distance

between the centres of the faces, it will also be the distance between the faces. Squaring the expression for z and then taking the root, we get $|z| = \sqrt{\frac{n+1}{2(n-k)(k+1)}}$.

1379. *Hint.* Find α from the condition $(x - \alpha e, e) = 0$. When proving uniqueness, form the scalar product of both sides of the equation $\alpha_1 \cdot e + z_1 = \alpha_2 e + z_2$ into e.

1380. *Hint.* Consider the scalar square of the vector $y = x - \sum_{i=1}^{k} \alpha_i e_i$, where $\alpha_i = (x, e_i)$, and apply the properties (c) and (d) of the preceding problem.

1381. *Hint.* First method: consider the scalar square $(x + ty, x + ty)$ as a nonnegative quadratic trinomial in t. Second method: for $y \neq 0$ represent x as $x = \alpha y + z$, where $(y, z) = 0$, show that $(x, x) \geqslant \alpha^2 (y, y)$, equality occurring if and only if $x = \alpha y$, and determine

$$(x, y)^2 = \alpha^2 (y, y)(y, y) \leqslant (x, x)(y, y).$$

Third method: apply the inequality of problem 503 to the coordinates of the vectors x and y in an orthonormal basis.

1382. *Hint.* First method: consider the scalar square $(x + ty, x + ty)$, where $t = s(x, y)$ as a nonnegative quadratic trinomial in s (s real). Second method: for $y \neq 0$ put $x = \alpha y + z$, where α is a complex number and $(y, z) = 0$; show that $(x, x) \geqslant \alpha\bar{\alpha} (y, y)$, equality occurring if and only if $x = \alpha y$, and find out that

$$(x, y)(y, x) = \alpha\bar{\alpha} (y, y)(y, y) \leqslant (x, x)(y, y).$$

Third method: apply the inequality of problem 505 to the coordinates of the vectors x and y in an orthonormal basis.

1384. $\left(\int_a^b f(x) g(x)\, dx\right)^2 \leqslant \int_a^b [f(x)]^2\, dx \cdot \int_a^b [g(x)]^2\, dx.$

1385. $AB = BC = AC = 6$; $\angle A = \angle B = \angle C = 60°$.

1386. $AB = 5$; $BC = 10$; $AC = 5\sqrt{3}$; $\angle A = 90°$; $\angle B = 60°$; $\angle C = 30°$.

1389. *Hint.* If the edges are given by the vectors $a_1, a_2, \ldots, a_n$, then consider the expression $|a_1 + a_2 + \ldots + a_n|^2$.

1390. *Hint.* Consider the expression $|x+y|^2 + |x - y|^2$.

1393. When n is odd, there are no orthogonal diagonals; when $n = 2k$ the desired number is equal to $\frac{1}{2} C_n^k = C_{2k-1}^{k-1}$.

1394. $a \cdot \sqrt{n}$; $\lim\limits_{n \to \infty} a\sqrt{n} = \infty$.

1395. $\varphi_n = \arccos \frac{1}{\sqrt{n}}$; $\lim\limits_{n \to \infty} \varphi_n = \frac{\pi}{2}$; $\varphi_4 = 60°$.

1396. $R = \frac{a\sqrt{n}}{2}$. $R < a$ for $n = 1, 2, 3$; $R = a$ for $n = 4$, and $R > a$ for $n > 4$.

1398. *Hint.* Show that the origin of a diagonal may be joined to any other vertex by a chain of edges and make use of the preceding problem.

1399. *Hint.* Take advantage of problem 1379.

1400. *Solution.* $\cos(x, y)=\dfrac{(x, y)}{|x|\cdot|y|}=\dfrac{(y, y)}{|x|\cdot|y|}=\dfrac{|y|}{|x|}$.

$\cos(x, y')=\dfrac{(x, y')}{|x|\cdot|y'|}=\dfrac{(y, y')}{|x|\cdot|y'|}=\dfrac{|y|\cdot|y'|}{|x|\cdot|y'|}\cos(y, y')\leqslant\dfrac{|y|}{|x|}=$
$=\cos(x, y)$.

Equality is possible if and only if $\cos(y, y')=1$, that is, according to problem 1399, when $y'=\alpha\cdot y$ for $\alpha>0$.

1401. $\arccos\sqrt{\dfrac{k}{n}}$. 1402. 60°. 1403. 30°.

1405. $\arccos\dfrac{2}{3}$. *Hint.* Let a_i be a vector from A_0 to A_i $(i=1, 2, 3, 4)$. Consider the two vectors $a_1t_1+a_2t_2$ and $a_3t_3+a_4t_4$; show that the square of the cosine of the angle between them is equal to $\dfrac{(t_1+t_2)^2(t_3+t_4)^2}{4(t_1^2+t_1t_2+t_2^2)(t_3^2+t_3t_4+t_4^2)}$ and find the maximum of the function $(t_1+t_2)^2$, provided that $t_1^2+t_1t_2+t_2^2=1$.

1406. 45°. *Hint.* Seek the minimum of the angles of the vectors of the second plane with their orthogonal projections on the first plane.

1407. *Hint.* Show that each of the systems $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ is a basis of the subspace L_k spanned by the vectors $e_1, \ldots, e_k$, that $(f_i, g_j)=0$ for $i\neq j$, and, finally, that all coefficients $c_1, c_2, \ldots, c_{k-1}$ are zero in the equation $g_k=c_1f_1+\ldots+c_kf_k$.

1408. *Hint.* Set $(x^2-1)^k=u_k(x)$ and verify that $u_k^{(j)}(\pm 1)=0$ for $j<k$; then integrate $\int\limits_{-1}^{+1}u_k^{(k)}(x)\,x^j\,dx$ by parts several times until the factor of the form x^s vanishes under the integral sign and show that this integral is equal to zero for $j=0, 1, \ldots, k-1$. From this derive the required equality: $\int\limits_{-1}^{+1}P_j(x)\,P_k(x)\,dx=0$ for $j\neq k$.

1409. $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\dfrac{1}{2}(3x^2-1)$, $P_3(x)=\dfrac{1}{2}(5x^3-3x)$,

$P_4(x)=\dfrac{1}{8}(35x^4-30x^2+3)$, $P_k(x)=\dfrac{1}{2^k k!}\sum\limits_{j=0}^{k}(-1)^{k-j}C_k^j\dfrac{(2j)!}{(2j-k)!}x^{2j-k}=$

$=\sum\limits_{j=0}^{k}(-1)^{k-j}\dfrac{1\cdot3\cdot5\cdot\ldots\cdot(2j-1)}{(k-j)!\,(2j-k)!\,2^{k-j}}x^{2j-k}$. In these expressions, drop all terms with negative exponents of x.

1410. $\sqrt{\frac{2}{2k+1}}$. *Solution.* Set $(x^2-1)^k = u_k(x)$ and compute the scalar square (P_k, P_k). Integrating by parts, we find

$$\int_{-1}^{+1} u_k^{(k)}(x) \cdot u_k^{(k)}(x)\,dx = -\int_{-1}^{+1} u_k^{(k-1)}(x)\, u_k^{(k+1)}(x)\,dx$$

$$\ldots = (-1)^k \int_{-1}^{+1} u_k(x)\, u_k^{(2k)}(x)\,dx = (2k)! \int_{-1}^{+1} (1-x)^k (1+x)^k\,dx.$$

Again integrating by parts, we find

$$\int_{-1}^{+1} (1-x)^k (1+x)^k\,dx = \frac{k}{k+1} \int_{-1}^{+1} (1-x)^{k-1} (1+x)^{k+1}\,dx$$

$$\ldots = \frac{k!}{(k+1)(k+2)\ldots(2k)} \int_{-1}^{+1} (1+x)^{2k}\,dx = \frac{(k!)^2\, 2^{2k+1}}{(2k)!\,(2k+1)},$$

whence

$$(P_k, P_k) = \frac{1}{2^{2k}(k!)^2}(2k)! \frac{(k!)^2\, 2^{2k+1}}{(2k)!\,(2k+1)} = \frac{2}{2k+1}.$$

1411. $P_k(1) = 1$. *Hint.* Apply to the expression $P_k(x) = \frac{1}{2^k k!} \times \frac{d^k}{dx^k}[(x+1)^k (x-1)^k]$ the Leibniz rule for differentiating a product.

1412. $P_k(x) = C_k f_k(x)$, where $C_k = \frac{(2k)!}{2^k (k!)^2} = \frac{C_{2k}^k}{2^k} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2k-1}{k!}$ is the leading coefficient of the polynomial $P_k(x)$ $(k = 0, 1, \ldots, n)$. *Hint.* Make use of the problems 1407, 1408, 1409.

1414. *Hint.* Setting $x^n = [x^n - f(x)] + f(x)$, show that a minimum is attained if and only if $f(x)$ is an orthogonal component for x^n relative to the subspace of polynomials of degree $\leqslant n-1$ (problem 1373), and apply the problems 1410, 1412, and 1413.

1415. $g(a_1, a_2)$ is equal to the square of the area of a parallelogram constructed on the vectors a_1, a_2; $g(a_1, a_2, a_3)$ is equal to the square of the volume of a parallelepiped constructed on the vectors a_1, a_2, a_3.

1416. *Hint.* Consider a system of homogeneous linear equations with determinant $g(a_1, a_2, \ldots, a_k)$.

1417. *Hint.* Seek the dual basis from the relations

$$f_j = \sum_{k=1}^{n} c_{jk} e_k \qquad (j = 1, 2, \ldots, n).$$

1418. (a) $T = (S')^{-1}$, (b) $T = (\bar{S}')^{-1}$. The prime stands for the transpose, the bar stands for replacing elements by complex conjugate elements.

1419. *Hint.* First method: show that the Gram determinant $g(a_1, \ldots, a_k)$ is equal to the square of the modulus of the determinant made up of the coordinates of the vectors $a_1, \ldots, a_k$ in any orthonormal basis of a k-dimensional subspace containing those vectors.

Second method: show that the nonnegative quadratic form $(a_1x_1 + \ldots + a_kx_k, a_1x_1 + \ldots + a_kx_k)$ in $x_1, \ldots, x_k$ is positive definite if and only if the vectors $a_1, \ldots, a_k$ are linearly independent.

Third method: using the invariance of the Gram determinant when orthogonalizing vectors (problem 1415), show that if the vectors $b_1, \ldots, b_k$ are obtained from $a_1, \ldots, a_k$ by the process of orthogonalization, then $g(a_1, \ldots, a_k) = |b_1|^2 \ldots |b_k|^2$, and apply problem 1413.

1421. $\dfrac{1}{C_{2n}^n \sqrt{2n+1}}$. *Hint.* First method: noting that the desired distance is equal to the length of the orthogonal component of the vector $-x^n$ relative to the subspace of polynomials of degree not exceeding $n-1$, apply the preceding problem and also problem 418.

Second method (it dispenses with problem 418): the desired distance yields the minimum of the integral $\int_0^1 [f(x)]^2\,dx$, where $f(x)$ is an nth-degree polynomial with leading coefficient unity. This enables one, by changing the limits of integration, to reduce the problem to the appropriate extremal property of the Legendre polynomial (problem 1414).

1422. *Hint.* Apply problems 1413 and 1415.

1423. $|D^2| \leqslant \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2$, equality occurring if and only if either $\sum_{k=1}^{n} a_{ik}\bar{a}_{jk} = 0$ $(i \neq j;\ i, j = 1, 2, \ldots, n)$ or the determinant D contains a zero row.

1424. *Hint.* Introduce into the vector space R_n the scalar product $(x, y) = \sum_{i,j=1}^{n} a_{ij}x_i\bar{y}_j$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the coordinates of x and y respectively in some basis $e_1, \ldots, e_n$ of R_n; show that $D_f = g(e_1, \ldots, e_n)$ and apply problem 1422.

1425. *Hint.* Use problem 1210 (d).

1426. *Hint.* Use the properties of Hermitian forms similar to the properties of the real forms indicated in problem 1210 (see, for example, F. R. Gantmakher's "The Theory of Matrices", Chelsea, New York, 1959).

1427. *Hint.* Make use of the reasoning in the first and third methods given in the answer to problem 1419.

1428. *Hint.* Apply problem 1422.

1429. *Solution.* Let the process of orthogonalization carry the vectors $a_1, \ldots, a_k, b_1, \ldots, b_l$ into the vectors $c_1, \ldots, c_k, d_1, \ldots, d_l$, and the vectors $b_1, \ldots, b_l$ into the vectors $e_1, \ldots, e_l$. The orthogonal component of the vector e_i relative to the subspace L_i spanned by $a_1, \ldots, a_k, b_1, \ldots, b_{i-1}$ coincides with d_i.

Indeed, $b_l = y_l + e_l$, where y_l can be linearly expressed n terms of $b_1, \ldots, b_{l-1}$, and e_l is orthogonal to these vectors. $e_l = y_l' + z$, where $y_l' \in L_l$ and z is orthogonal to L_l, but then $b_l = (y_l + y_l') + z$, where $y_l + y_l' \in L_l$ and z is orthogonal to L_l. Hence, by problem 1413, $z_l = d_l$ and $|d_l| \leqslant |e_l|$, equality definitely occurring under condition (2) because $e_l = b_l - y_l$ is expressed in terms of $b_1, \ldots, b_l$ and, hence, is orthogonal to $a_1, \ldots, a_k$; $b_1, \ldots, b_{l-1}$. By problem 1415 we have

$$g(a_1, \ldots, a_k, b_1, \ldots, b_l) = (|c_1|^2 \cdot \ldots \cdot |c_k|^2)$$

$$\times (|d_1|^2 \cdot \ldots \cdot |d_l|^2) \leqslant (|c_1|^2 \cdot \ldots \cdot |c_k|^2) \cdot (|e_1|^2 \cdot \ldots \cdot |c_l|^2)$$

$$= g(a_1, \ldots, a_k)\, g(b_1, \ldots, b_l),$$

which proves inequality (1). Under the condition (2), $|d_l| = |e_l|$ $(l = 1, 2, \ldots, l)$ and inequality (1) becomes an equality. If $a_1, \ldots, a_k$ or $b_1, \ldots, b_l$ are linearly dependent, then the right member of inequality (1) vanishes, but since the left member is nonnegative, we again obtain an equality.

Conversely, let the inequality (1) become an equality. Then by what has already been demonstrated $|c_1|^2 \ldots |c_k|^2 \cdot |d_1|^2 \ldots \ldots |d_l|^2 = |c_1|^2 \ldots |c_k|^2 \cdot |e_1|^2 \ldots |e_l|^2$, whence there either exists an $i \leqslant k$ such that $|c_l| = 0$, that is $a_1, \ldots, a_k$ are linearly dependent, or there exists an $l \leqslant l$ such that $|d_l| = |e_l| = 0$, that is, $b_1, \ldots, b_l$ are linearly dependent, or $|d_i| = |e_i|$ $(i = 1, 2, \ldots, l)$, whence it follows that all e_l and, hence, all b_i (as linear combinations of them) are orthogonal to $a_1, \ldots, a_k$, which means condition (2) is fulfilled.

1430. *Hint.* Use the preceding problem.

1431. *Hint.* Make use of problems 1425 and 1429.

1432. *Hint.* Make use of problems 1426 and 1429.

1433. *Solution.* (a) Let the numbers of system (1) be the distances of all possible pairs of vertices $M_0, M_1, \ldots, M_n$ of an n-dimensional simplex, and suppose

$$a_{ij} = \overline{M_i M_j} \quad (i, j = 0, 1, 2, \ldots n;\ i > j). \tag{2}$$

Denote by e_i the vector from M_0 to M_i $(i = 1, 2, \ldots, n)$. We then have

$$\begin{aligned} a_{i0}^2 &= (e_i, e_i) && (i = 1, 2, \ldots, n); \\ a_{ij}^2 &= (e_i - e_j, e_i - e_j) && (i, j = 1, 2, \ldots, n;\ i > j). \end{aligned} \tag{3}$$

From these equalities we find the scalar products of the vectors $e_1, \ldots, e_n$:

$$\begin{aligned} (e_i, e_i) &= a_{i0} && (i = 1, 2, \ldots, n), \\ (e_i, e_j) &= \frac{a_{i0} + a_{j0} - a_{ij}}{2} && (i, j = 1, 2, \ldots, n;\ (i > j). \end{aligned} \tag{4}$$

Using these relations, we can write down the Gram matrix of the vectors $e_1, \ldots, e_n$:

$$\begin{Vmatrix} a_{10} & \frac{a_{20}+a_{10}-a_{21}}{2} & \ldots & \frac{a_{n0}+a_{10}-a_{n1}}{2} \\ \frac{a_{20}+a_{10}-a_{21}}{2} & a_{20} & \ldots & \frac{a_{n0}+a_{20}-a_{n2}}{2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{a_{n0}+a_{10}-a_{n1}}{2} & \frac{a_{n0}+a_{20}-a_{n2}}{2} & \ldots & a_{n0} \end{Vmatrix}. \quad (5)$$

Since the points $M_0, M_1, \ldots, M_n$ do not lie in an $(n-1)$-dimensional manifold, the vectors $e_1, e_2, \ldots, e_n$ are linearly independent. Denote by D_k the corner minor of order k of the matrix (5) and apply problem 1419 to get

$$D_k > 0 \quad (k = 1, 2, \ldots, n). \quad (6)$$

Thus, conditions (6) are the necessary conditions for the numbers of system (1) to be the distances of the vertices of an n-dimensional simplex. We now show that these conditions are sufficient. When conditions (6) are fulfilled, the matrix (5) is the Gram matrix of a linearly independent set of vectors $e_1, \ldots, e_n$ [see problem 1352 or 1210 (c)]. Thus equalities (4) hold, whence follow equalities (3). Taking M_0 for the origin of coordinates and M_i for the terminus of the vector e_i $(i = 1, 2, \ldots, n)$, we see that equalities (2) hold, which is what we set out to prove.

In the case (b), a necessary and sufficient condition is that all principal minors (not only corner minors) of matrix (5) be nonnegative. The necessity is demonstrated in the same way as in the case (a) with the sole difference that the vectors $e_1, \ldots, e_n$ may be linearly dependent. Sufficiency is demonstrated with the aid of problem 1425 or 1210 (d).

1434. $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

1435. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ if e_1 goes into e_2 and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ if e_2 goes into e_1.

1436. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **1437.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. **1438.** $\begin{Vmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{Vmatrix}$.

1439. The first k elements of the main diagonal of the transformation matrix are equal to unity, all other elements being zero.

1440. The ith column of the transformation matrix contains the coordinates of the vector b_i in the basis $a_1, \ldots, a_n$.

1441. φ is linear. $A_\varphi = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$. **1442.** φ is not linear.

1443. φ is not linear. **1444.** φ is linear. $A_\varphi = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

1445. $\begin{pmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -1 & 0 \end{pmatrix}$. **1446.** $\frac{1}{3}\begin{pmatrix} -6 & 11 & 5 \\ -12 & 13 & 10 \\ 6 & -5 & -5 \end{pmatrix}$.

1448. In the basis e_1, e_2, e_3 the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix};$$

in the basis b_1, b_2, b_3 the matrix

$$\begin{pmatrix} \frac{20}{3} & -\frac{5}{3} & 5 \\ -\frac{16}{3} & \frac{4}{3} & -4 \\ 8 & -2 & 6 \end{pmatrix}.$$

1449. (a) Premultiplication yields (b) postmultiplication yields

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}; \qquad \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix}.$$

1450. (a) $\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$; (b) $\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$.

1451. The ith and jth rows and the ith and jth columns in the matrix are interchanged.

1452. (a) $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & 5 & 1 \\ 3 & -1 & 0 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}$; (b) $\begin{pmatrix} -2 & 0 & 1 & 0 \\ 1 & -4 & -8 & -7 \\ 1 & 4 & 6 & 4 \\ 1 & 3 & 4 & 7 \end{pmatrix}$.

1453. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. **1454.** $\begin{pmatrix} 1 & 2 & 2 \\ 3 & -1 & -2 \\ 2 & -3 & 1 \end{pmatrix}$.

1457. $\begin{pmatrix} 44 & 44 \\ -29\frac{1}{2} & -25 \end{pmatrix}$. **1458.** $\begin{pmatrix} 109 & 93 \\ 34 & 29 \end{pmatrix}$. **1461.** n^2.

1464. *Hint.* Use the preceding problem and also the definition of the function of a matrix (problem 1148).

1465. $\lambda_1 = \lambda_2 = \lambda_3 = -1$. The eigenvectors are of the form $c\,(1, 1, -1)$, where $c \neq 0$.

1466. $\lambda_1 = \lambda_2 = \lambda_3 = 2$. The eigenvectors have the form $c_1(1, 2, 0) + c_2(0, 0, 1)$, where c_1 and c_2 are not simultaneously zero.

1467. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$. The eigenvectors for the value 1 are of the form $c\,(1, 1, 1)$, for $\lambda = 0$ of the form $c\,(1, 2, 3)$, where $c \neq 0$.

1468. $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The eigenvectors are of the form $c\,(3, 1, 1)$, where $c \neq 0$.

1469. $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -1$. The eigenvectors for $\lambda = 3$ are of the form $c\,(1, 2, 2)$, for $\lambda = -1$ of the form $c\,(1, 2, 1)$, where $c \neq 0$.

1470. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. The eigenvectors for $\lambda = 1$ are of the form $c_1(2, 1, 0) + c_2(1, 0, -1)$ and for $\lambda = -1$ of the form $c\,(3, 5, 6)$, where c_1 and c_2 are not simultaneously zero and $c \neq 0$.

1471. $\lambda_1 = 1$, $\lambda_2 = 2 + 3i$, $\lambda_3 = 2 - 3i$. The eigenvectors for $\lambda = 1$ are of the form $c\,(1, 2, 1)$, for $\lambda = 2 + 3i$ of the form $c\,(3 - 3i, 5 - 3i, 4)$, for $\lambda = 2 - 3i$ of the form $c\,(3 + 3i, 5 + 3i, 4)$.

1472. $\lambda_1 = \lambda_2 = 1$; $\lambda_3 = \lambda_4 = 0$. The eigenvectors for $\lambda = 1$ are of the form $c\,(0, 0, 0, 1)$, for $\lambda = 0$ of the form $c_1(0, 1, 0, 0) + c_2(0, 0, 1, 0)$, where $c \neq 0$ and c_1 and c_2 are not simultaneously zero.

1473. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$. The eigenvectors for $\lambda = 1$ are of the form $c_1(1, 0, 1, 0) + c_2(0, 0, 0, 1)$, for $\lambda = 0$ of the form $c_1(0, 1, 0, 0) + c_2(0, 0, 1, 0)$, where the numbers c_1 and c_2 are not simultaneously zero.

1474. $\lambda = 2$. The eigenvectors are of the form $c_1(1, 1, -1, 0) + c_2(1, 1, 0, 1)$, where c_1 and c_2 are not simultaneously zero. **1477.** *Hint.* Make use of problems 820 and 1476.

1479. $a_1 = (1, 1, 1)$, $a_2 = (1, 1, 0)$, $a_3 = (1, 0, -3)$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

1480. The matrix cannot be reduced to diagonal form.

1481. $a_1=(1,\ 1,\ 0,\ 0)$, $a_2=(1,\ 0,\ 1,\ 0)$, $a_3=(1,\ 0,\ 0,\ 1)$, $a_4=(1,\ -1,\ -1,\ -1)$; $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

1482. The matrix cannot be reduced to diagonal form.

1483. $a_1=(1,\ 0,\ 0,\ 1)$, $a_2=(0,\ 1,\ 1,\ 0)$, $a_3=(0,\ -1,\ 1,\ 0)$, $a_4=(-1,\ 0,\ 0,\ 1)$; $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

1484. For example,

$$T=\begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & -1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where along the secondary diagonal the elements above the main diagonal are -1 and below are $+1$; when n is odd, both $+1$ and -1 can reside at the intersection of the diagonals. The diagonal matrix B has, above the main diagonal, $n/2$ elements (for even n) and $\frac{n+1}{2}$ elements (for odd n) equal to $+1$, the remaining elements being equal to -1.

Hint. Consider a linear transformation φ (of n-dimensional space) having the matrix A in a certain basis, find the basis consisting of the eigenvectors of the transformation, and apply problem 1477.

1486. Any two elements α_k and α_{n-k+1} equidistant from the ends of the secondary diagonal either must both be nonzero or must both vanish.

Hint. Consider a linear transformation φ (of n-dimensional space) that corresponds to the matrix A in some basis, and then apply problems 1117, 1132, 1484.

1487. The sole eigenvalue is $\lambda = 0$; the eigenvectors are polynomials of zero degree.

1491. *Solution.* First prove the equality

$$\dim L = \dim \varphi L + \dim L_0, \qquad (1)$$

where L_0 is the intersection of L and the kernel $\varphi^{-1}0$ of the transformation φ. To do this, first complete the basis $a_1, a_2, \dots, a_k$ of the subspace L_0 (using the vectors $b_1, b_2, \dots, b_l$) to form a basis of L (for $L_0=0$ the vectors a_i are absent, for $L_0=L$ the vectors b_l are absent). The vectors $\varphi b_1, \varphi b_2, \dots, \varphi b_l$ form a basis of φL. Indeed, if $y \in \varphi L$, then $y=\varphi x$, where $x \in L$. If $x = \sum_{i=1}^{k} \alpha_i a_i + \sum_{i=1}^{l} \beta_i b_i$, then $y=\varphi x = \sum_{i=1}^{l} \beta_i \varphi b_i$ since $\varphi a_i = 0$

$(i = 1, 2, \ldots, k)$. The vectors $\varphi b_1, \varphi b_2, \ldots, \varphi b_l$ are linearly independent since $\sum_{i=1}^{l} \beta_i \varphi b_i = 0$ implies $\sum_{i=1}^{l} \beta_i b_i \in L_0$, whence $\sum_{i=1}^{l} \beta_i b_i = \sum_{i=1}^{k} \alpha_i a_i$ and, hence, $\beta_i = 0$ $(i = 1, 2, \ldots, l)$.

Thus, $\dim L = l + k = \dim \varphi L + \dim L_0$.

(a) From (1) we find, by virtue of $L_0 \subset \varphi^{-1}0$,

$$\dim L = \dim \varphi L + \dim L_0 \leqslant \dim \varphi L + \text{nullity } \varphi;$$

$$\dim L - \text{nullity } \varphi \leqslant \dim \varphi L$$

Furthermore, $\dim \varphi L = \dim L - \dim L_0 \leqslant \dim L$.

(b) Set $\varphi^{-1}L = L'$. Since $0 \in L$, then $\varphi^{-1}0 \subset \varphi^{-1}L = L'$ and $L' \cap \varphi^{-1}0 = \varphi^{-1}0$. Applying (1) with L' substituted for L, we get

$$\dim L' = \dim \varphi L' + \text{nullity } \varphi \qquad (2)$$

Since $\varphi L' \subset L$, it follows that $\dim \varphi L' \leqslant \dim L$ and by (2) $\dim L' \leqslant \dim L + \text{nullity } \varphi$, which thus proves the second of the inequalities of (b).

Now let us show that $\varphi L' = L \cap \varphi R_n$.

Since $\varphi L' \subset L$ and $\varphi L' \subset \varphi R_n$, then $\varphi L' \subset L \cap \varphi R_n$. If $x \in L \cap \varphi R_n$, then $x = \varphi x'$, where $x' \in \varphi^{-1}L = L'$, that is, $x \in \varphi L'$, whence $L \cap \varphi R_n \subset \varphi L'$.

Since $\dim (L + \varphi R_n) \leqslant n$, we get, using the relationship between the dimension of a sum and an intersection of subspaces (problem 1316),

$$\dim \varphi L' = \dim (L \cap \varphi R_n) = \dim L + \dim \varphi R_n - \dim (L + \varphi R_n)$$
$$\geqslant \dim L + \dim \varphi R_n - n = \dim L - \text{nullity } \varphi.$$

From this, via (2), we obtain

$$\dim L' = \dim \varphi L' + \text{nullity } \varphi \geqslant (\dim L - \text{nullity } \varphi) + \text{nullity } \varphi = \dim L$$

and this completes the proof of the first of the inequalities of (b).

1492. *Hint.* Consider the transformations φ and ψ of the space R_n with matrices A and B and apply the preceding problem to the subspace $L = \psi R_n$.

1494. The sole eigenvalue is $\lambda = 1$. The eigenvectors have the form $c_1 (a_1 + 2a_2) + c_2 (a_2 + a_3 + 2a_4)$, where c_1, c_2 are not simultaneously zero.

1495. *Hint.* Consider the matrix of the transformation φ in a basis whose first vectors are linearly independent eigenvectors of φ associated with λ_0. An alternative approach is by using problem 1074.

1501. The zero subspace and the subspace L_k, which consist of all polynomials of degree $\leqslant k$ $(k = 0, 1, 2, \ldots, n)$.

1503. The invariant subspaces are: the zero subspace and any subspace spanned by an arbitrary subsystem of vectors of the basis $a_1, a_2, \ldots, a_n$; their total number is 2^n.

Hint. Use problems 1495 and 1502 to show that any nonzero invariant subspace L has a basis which is a subset of the vectors $a_1, a_2, \ldots, a_n$.

1504. A straight line with basis vector $a_1 = (2, 2, -1)$, any straight line of plane L with basis vectors $a_2 = (1, 1, 0)$ and $a_3 = (1, 0, -1)$, that is, of the plane $x_1 - x_2 + x_3 = 0$, the plane L itself, any plane passing through the vector a_1, the whole space, and the zero subspace.

1505. A straight line with basis vector $(1, -2, 1)$, a plane with basis vectors $(1, 1, 1)$ and $(1, 2, 3)$, that is the plane with the equation $x_1 - 2x_2 + x_3 = 0$, the whole space, and the zero subspace.

1509. $\lambda_1 = 1$, $\lambda_{2,3} = 0$. The root subspaces consist of vectors for $\lambda_1 = 1$: $c(1, 1, 1)$; for $\lambda_{2,3} = 0$: $c_1(1, 1, 0) + c_2(1, 0, -3)$.

1510. $\lambda_1 = 3$, $\lambda_{2,3} = -1$. The root subspaces consist of the vectors for $\lambda = 3$: $c(1, 2, 2)$; for $\lambda_{2,3} = -1$; $c_1(1, 1, 0) + c_2(1, 0, -1)$.

1511. $\lambda_{1,2,3} = -1$. The entire space is a root subspace.

1512. $\lambda_{1,2} = 2$, $\lambda_{3,4} = 0$. The root subspaces consist of the vectors:

$$\text{for } \lambda_{1,2} = 2\text{: } c_1(1, 0, 1, 0) + c_2(1, 0, 0, 1);$$

$$\text{for } \lambda_{3,4} = 0\text{: } c_1(1, 0, 0, 0) + c_2(0, 1, 0, 1).$$

1515. (a) Any number α is an eigenvalue. The corresponding eigenvectors are of the form $ce^{\alpha x}$, where $c \neq 0$;

(b) the root subspace corresponding to the number α consists of all functions of the form $(a_0 + a_1x + \ldots + a_nx^n)e^{\alpha x}$, where $a_0, a_1, \ldots, a_n$ are any numbers and n is any nonnegative integer.

1517. If the minimal polynomial $g(\lambda) = \lambda^n - c_n\lambda^{n-1} - c_{n-1}\lambda^{n-2} - \ldots - c_1$, then the transformation matrix φ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & c_1 \\ 1 & 0 & 0 & \ldots & 0 & c_2 \\ 0 & 1 & 0 & \ldots & 0 & c_3 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & c_n \end{pmatrix}.$$

1518. The Jordan submatrix

$$\begin{pmatrix} \alpha & 1 & 0 & \ldots & 0 \\ 0 & \alpha & 1 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & \alpha \end{pmatrix}.$$

1520. For $n = 2$, rotation of the plane through the angle $\alpha \neq k\pi$ does not possess invariant straight lines. The subspace L has a straight line of fixed points if and only if the transformation φ_1 induced by the transformation φ on L has an eigenvalue equal to unity.

1523. *Hint.* When proving assertion (a), make use of the equality $1 = h_1(\lambda)u_1(\lambda) + h_2(\lambda)u_2(\lambda)$, where $u_1(\lambda)$ and $u_2(\lambda)$ are polynomials in λ; (b) derive the proof from (a).

1524. $g(\lambda) = (\lambda - 1)^2(\lambda - 2)$. $R_3 = L_1 \dotplus L_2$, where L_1 has the basis $f_1 = e_1$, $f_2 = e_2 - e_3$ and L^2 has the basis $f_3 = e_2$.

Hint. Use problem 1485 to find $g(\lambda)$.

1525. $g(\lambda) = (\lambda - 2)(\lambda - 3)$. $R_3 = L_1 \dot{+} L_2$, where L_1 has, for example, the basis $f_1 = e_1 + e_2$, $f_2 = e_1 - e_3$ and L_2 the basis $2e_1 - 5e_2 - 6e_3$.

1526. $g(\lambda) = [\lambda - (a, a)]\lambda$. $R_n = L_1 \dot{+} L_2$, where L_1 is spanned by the vector a and L_2 is formed by all vectors orthogonal to a.

1527. If λ_0 is an eigenvalue of φ, then the Jordan form consists of a single block of order n with λ_0 on the diagonal.

1529. *Solution.* (A). Set $f_i(\lambda) = \dfrac{f(\lambda)}{(\lambda - \lambda_i)^{k_i}}$ $(i = 1, 2, \ldots, s)$. The polynomials $f_1(\lambda)$ are relatively prime. Hence, there exist polynomials $h_i(\lambda)$ $(i = 1, 2, \ldots, s)$ such that $1 = \sum_{i=1}^{s} f_i(\lambda) h_i(\lambda)$, whence for any vector x

$$x = \sum_{i=1}^{s} x_i, \tag{1}$$

where $x_i = f_i(\varphi) \cdot h_i(\varphi) x \in P_i$ since $(\varphi - \lambda_i \varepsilon)^{k_i} x_i = f(\varphi) h_i(\varphi) x = 0$ by the Hamilton-Cayley theorem, by virtue of which $f(\varphi) = 0$.

It suffices to prove the uniqueness of the expansion (1) for $x = 0$. Applying the transformation $f_i(\varphi)$ to both sides of the equality $\sum_{i=0}^{s} x_i = 0$, we get $f_i(\varphi) x_i = 0$ since $f_i(\varphi) x_j = 0$ for $j \neq i$. Furthermore, $f_i(\lambda)$ and $(\lambda - \lambda_i)^{k_i}$ are relatively prime. Hence, there exist polynomials $p(\lambda)$ and $q(\lambda)$ such that

$$1 = p(\lambda) f_i(\lambda) + q(\lambda)(\lambda - \lambda_i)^{k_i},$$

whence

$$x_i = p(\varphi) f_i(\varphi) x_i + q(\varphi)(\varphi - \lambda_i \varepsilon)^{k_i} x_i = 0.$$

This proves that the space R_n is a direct sum of the subspaces P_i $(i = 1, 2, \ldots, s)$ and the construction of the desired basis has been reduced to the case (B). In the case of a minimal polynomial, the reasoning is similar (see problem 1523).

(B) It suffices to prove that the indicated construction is possible at every step (that is, the vectors that serve to complete the basis of R_h are linearly independent) and that the vectors of all constructed sets form a basis of the space R_n. To each set in the transformation matrix ψ there obviously corresponds a Jordan submatrix with zero, and in the transformation matrix $\varphi = \lambda_0 \varepsilon + \psi$ there corresponds a Jordan submatrix with λ_0 on the diagonal.

The possibility of construction at each step is proved inductively for $h = k, k - 1, \ldots, 1$. When $h = k$, the vectors of any basis of R_{k-1} together with the vectors of the height k that start the sets of the first step form, by construction, a basis of R_k. Let us suppose that the sets with initial vectors of height $\geqslant h + 1$ have already been constructed, and all vectors of height $h+1$ of the constructed sets $x_1, \ldots$

$\ldots, x_p$ together with the vectors $y_1, \ldots, y_q$ of any basis of R_h form a basis of R_{h+1}. We will show that the vectors of height h: $\psi x_1, \ldots, \psi x_p$ of the constructed sets together with any basis $z_1, \ldots, z_r$ for R_{h-1} are linearly independent. Let

$$\sum_{i=1}^{p} c_i \psi x_i + \sum_{j=1}^{r} d_j z_j = 0 \qquad (2)$$

Applying the transformation ψ^{h-1} to both sides of this equality, we get $\psi^h \sum_{i=1}^{p} c_i x_i = 0$. Therefore the vector $\sum_{i=1}^{p} c_i x_i$ belongs to R_h and can be linearly expressed in terms of its basis $y_1, \ldots, y_q$. From the linear independence of the vectors $x_1, \ldots, x_p, y_1, \ldots, y_q$ (as a basis of R_{h+1}) it follows that $c_i = 0$ $(i = 1, 2, \ldots, p)$. For this reason, from the equality (2) it follows that $d_j = 0$ $(j = 1, 2, \ldots, r)$; this proves that the vectors of height h of the earlier constructed sets together with the vectors of any basis of R_{h-1} may be completed to a basis of R_h. Taking the supplementary vectors (if they exist) for the initial vectors of new sets, we find that the assumption made above for R_{h+1} is now fulfilled for R_h, and the construction may be continued.

Suppose the indicated construction has been carried out for $h = k, k-1, \ldots, 1$ (actually, the basis of the entire space may result before we arrive at $h = 1$). Since R_0 does not have a basis, it follows from what has been proved that vectors of height 1 of the constructed sets form a basis for R_1; hence these vectors, together with vectors of height 2 of the constructed sets, form a basis of R_2, and so forth. Finally, vectors of height $\leqslant k-1$ of the constructed sets, together with vectors of height k of these sets, form a basis of R_k. In other words, the vectors of all constructed sets form a basis of the entire space.

(C) Let $C = A_J - \lambda_0 E$. Since the matrices B^h and C^h are similar, the rank of C^h is equal to r_h $(h = 0, 1, \ldots, k, k+1)$. To each Jordan submatrix of matrix A_J with λ_0 on the diagonal there corresponds in matrix C a block of the same order with zero on the diagonal. If D is such a block of order p, then the rank of block D^h for $h = 0, 1, 2, \ldots, p$ is equal to $p - h$, and for $h = p, p+1, \ldots, k, k+1$ is equal to zero. To the block with $\lambda_i \neq \lambda_0$ of matrix A_J there corresponds, in matrix C, a block with the number $\lambda_i - \lambda_0 \neq 0$ on the diagonal. The rank of any power of it is equal to its order. The rank of the matrix C^h is equal to the sum of the ranks of its blocks. Therefore, when passing from matrix C^{h-1} to matrix C^h the rank is lowered by exactly the number of blocks of matrix C with zero on the diagonal, the blocks having orders $\geqslant h$. Whence

$$\sum_{i=h}^{k} x_i = r_{h-1} - r_h \quad (h = 1, 2, \ldots, k) \qquad (3)$$

Subtracting from this a similar equality with $h+1$ substituted for h (for $h < k$), we obtain the relations (α) for $h = 1, 2, \ldots, k-1$. Since there are no blocks of orders exceeding k with λ_0 on the diagonal in matrix A_J, it follows that $r_k = r_{k+1}$, and for $h = k$ the relation (3)

yields $x_k = r_{k-1} - r_k = r_{h-1} - 2r_h + r_{h+1}$, which means the relation (α) holds for $h = k$ as well.

1530. $f_1=(1, 4, 3)$, $f_2=(1, 0, 0)$, $f_3=(3, 0, 1)$, $A_J=\begin{pmatrix}2&1&0\\0&2&0\\0&0&2\end{pmatrix}$.

1531. $f_1=(1, -3, -2)$, $f_2=(1, 0, 0)$, $f_3=(1, 0, 1)$, $A_J=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$.

1532. $f_1=(6, 6, -8)$, $f_2=(3, 1, 0)$, $f_3=(2, 1, -1)$; $A_J=\begin{pmatrix}-1&1&0\\0&-1&0\\0&0&0\end{pmatrix}$.

1533. $f_1=(3, 1, 1)$, $f_2=(1, 0, 0)$, $f_3=(5, 0, 1)$; $A_J=\begin{pmatrix}3&1&0\\0&3&0\\0&0&3\end{pmatrix}$.

1534. $f_1=(1, 1, 1, 1)$, $f_2=(-1, 0, 0, 0)$, $f_3=(1, 1, 0, 0)$, $f_4=(0, 0, -1, 0)$; $A_J=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$.

1535. $f_1=(-1, -1, -1, 0)$, $f_2=(2, 1, 0, 0)$, $f_3=(1, 0, 0, -1)$, $f_4=(3, 6, 7, 1)$; $A_J=\begin{pmatrix}2&1&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{pmatrix}$.

1536. For an even n:

$$f_1=e_1,\ f_2=e_3,\ f_3=e_5,\ \ldots,\ f_{\frac{n}{2}}=e_{n-1},$$

$$f_{\frac{n}{2}+1}=e_2,\ \ldots,\ f_n=e_n.$$

The matrix A_J consists of two Jordan submatrices of order $\frac{n}{2}$ with zero on the main diagonal. For an odd n:

$$f_1=e_1,\ f_2=e_3,\ f_3=e_5,\ \ldots,\ f_{\frac{n+1}{2}}=e_n,$$

$$f_{\frac{n+1}{2}+1}=e_2,\ \ldots,\ f_n=e_{n-1}.$$

The matrix A_J consists of two Jordan submatrices of orders $\frac{n+1}{2}$ and $\frac{n-1}{2}$ with zero on the main diagonal.

1537. φ is a reflection of the space R_n in some subspace L_1 parallel to some supplementary subspace L_2. In other words, R_n is a direct sum of L_1 and L_2 with $\varphi x = x$ if $x \in L_1$ and $\varphi x = -x$ if $x \in L_2$.

Hint. First method: for L_1 and L_2 take the collections of all x for which we respectively have $\varphi x = x$ and $\varphi x = -x$, and set

$$x = \frac{1}{2}(x + \varphi x) + \frac{1}{2}(x - \varphi x).$$

Second method: consider a basis in which the matrix $A\varphi$ has Jordan form.

1538. φ is a projection of the space R_n on some subspace L_1 parallel to some supplementary subspace L_2. In other words, R_n is a direct sum of L_1 and L_2 with $\varphi x = x$ if $x \in L_1$ and $\varphi x = 0$ if $x \in L_2$.

Hint. First method: for L_1 and L_2 take the collections of all x for which we respectively have $\varphi x = x$ and $\varphi x = 0$, and set $x = \varphi x + (x - \varphi x)$.

Second method: consider a basis in which the matrix $A\varphi$ is of Jordan form.

1539. (a) For the basis e_1, e_2, e_3, the transformation φ is defined thus:

$$\varphi e_1 = e_2, \quad \varphi e_2 = e_3, \quad \varphi e_3 = 0.$$

(b) $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$, $\varphi e_3 = 0$ (in the case of an orthonormal basis, φ is a projection—on the plane e_1, e_2—connected with a rotation of the plane through the angle $\frac{\pi}{2}$).

1541. $\begin{pmatrix} 3 & 6 \\ -1 & -3 \end{pmatrix}$.

1542. $\begin{pmatrix} -36 & -37 & -15 \\ 30 & 30 & 14 \\ 26 & 27 & 9 \end{pmatrix}$.

1543. $\begin{pmatrix} 4 & -2 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

1544. φ^* is a projection on the bisector of the second and fourth quadrants parallel to the y-axis.

1545. *Hint.* Show that if $z \in L_1^*$, $u \in L_2^*$, then $\varphi^* z = 0$, $\varphi^* u = u$.

1547. *Hint.* Show that an orthogonal complement to one-dimensional subspace that is invariant under the conjugate transformation φ^* is invariant to φ.

1548. *Hint.* Use the preceding problem to construct a chain of subspaces $R_n \supset L_{n-1} \supset \ldots \supset L_1$, where L_k is a k-dimensional subspace that is invariant under φ, and apply problem 1355.

1549. $3x_1 - 3x_2 + x_3 = 0$.

1553. *Hint.* Consider the equalities

$$(\varphi e_i, f_j) = (e_i, \varphi^* f_j) \quad (i, j = 1, 2, \ldots, n).$$

1554. *Hint.* Pass to an orthonormal basis.

1555. $A_1 = \begin{pmatrix} 128 & 313 & 454 \\ 36 & 117 & 145 \\ -61 & -197 & -245 \end{pmatrix}$.

1556. $A_1 = \begin{pmatrix} 5 & 5 & 3 \\ -5 & -7 & -2 \\ 3 & 6 & 0 \end{pmatrix}$.

1557. $A_1 = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$.

1558. $A_1 = \begin{pmatrix} 0 & -4 & 4 \\ 0 & -8 & 7 \\ -7 & -2 & 5 \end{pmatrix}$.

1562. For example, the tranformation φ that carries the vector $x = (x_1, x_2, \ldots, x_n)$, specified by coordinates in an orthonormal basis, into the vector $\varphi x = (|x_1|, x_2, \ldots, x_n)$ preserves scalar squares but is not linear.

Hint. To prove the linearity of φ, show that

$$(\varphi(ax + by) - a\varphi x - b\varphi y,\ \varphi(ax + by) - a\varphi x - b\varphi y) = 0$$

1563. (a) $UA^{-1} = A'U$, (b) $U\overline{A}^{-1} = A'U$.

1566. *Hint.* Show that there exists a vector $x_k' = x_k - \sum_{i=1}^{k-1} \alpha_i x_i$ (that may be zero) for which $(x_k', x_i) = 0$ $(i = 1, 2, \ldots, k-1)$ and that by putting $y_k' = y_k - \sum_{i=1}^{k-1} \alpha_i y_i$ we get systems of vectors $x_1, \ldots, x_{k-1}, x_k'$ and $y_1, \ldots, y_{k-1}, y_k'$ with equal Gram matrices; then apply the method of mathematical induction.

1567. *Hint.* Use problem 1499.

1569. *Solution.* (a) If $\varphi z = \lambda z$, then $(\varphi z, \varphi z) = \lambda\bar{\lambda}(z, z) = (z, z)$, whence $\lambda\bar{\lambda} = 1$ and $|\lambda| = 1$.

(b) Let $\varphi z_1 = \lambda_1 z_1$, $\varphi z_2 = \lambda_2 z_2$, $\lambda_1 \neq \lambda_2$. Then $(z_1, z_2) = (\varphi z_1, \varphi z_2) = \lambda_1\bar{\lambda}_2(z_1, z_2)$, whence, multiplying by λ_2 and taking into account that $\lambda_2\bar{\lambda}_2 = 1$, we find $\lambda_1(z_1, z_2) = \lambda_2(z_1, z_2)$ and, hence, $(z_1, z_2) = 0$.

(c) Let X and Y be the columns of the coordinates x and y. Passing to coordinates in the equation $\varphi(x + yi) = (\alpha + \beta i)(x + yi)$, we get $AX + AYi = (\alpha X - \beta Y) + (\beta X + \alpha Y)i$, whence, equating the real and imaginary parts, we obtain $AX = \alpha X - \beta Y$; $AY = \beta X + \alpha Y$, and that proves the equations (1). Multiplying termwise the first of equations (1) by itself and applying the relation $\alpha^2 + \beta^2 = 1$, we get $(\varphi x, \varphi x) = (x, x) = (\alpha^2 + \beta^2)|x|^2 = \alpha^2|x|^2 + \beta^2|y|^2 - 2\alpha\beta(x, y)$. Multiplying together equations (1), we get

$$(\varphi x, \varphi y) = (x, y) = (\alpha^2 + \beta^2)(x, y) = \alpha\beta(|x|^2 - |y|^2) + (\alpha^2 - \beta^2)(x, y).$$

Finally, cancelling out β, we obtain the following system of equations for $|x|^2 - |y|^2$ and (x, y):

$$\beta(|x|^2 - |y|^2) + 2\alpha(x, y) = 0;$$
$$\alpha(|x|^2 - |y|^2) - 2\beta(x, y) = 0.$$

Since the determinant of the system is nonzero, it follows that

$$|x|^2 - |y|^2 = 0 \text{ and } (x, y) = 0.$$

(d) If φ has a real eigenvalue, then there is a one-dimensional invariant subspace, otherwise we pass to a unitary space. Namely, we take the orthonormal basis $e_1, \ldots, e_n$ in the unitary space R'_n. The vectors of R'_n that have real coordinates in that basis form a Euclidean space R_n embedded in R'_n. The transformation φ has, in the basis $e_1, \ldots, e_n$, a real orthogonal matrix A. In the given basis, this matrix determines the unitary transformation φ' which coincides with φ on R_n. The transformation φ' has an eigenvalue $\alpha + \beta i$, where $\beta \neq 0$. If $x + yi$ is the corresponding eigenvector and x, y are vectors with real coordinates, then equalities (1) hold. Thus, the subspace of the space R_n, which subspace is spanned by the vectors x and y, is invariant under φ.

1570. (a) For any unitary matrix A there exists a unitary matrix Q such that the matrix $B = Q'AQ$ is a diagonal matrix with elements on the diagonal equal to unity in modulus.

(b) The space R_n is a direct sum of pairwise orthogonal one-dimensional and two-dimensional subspaces that are invariant under φ. The transformation φ leaves the vectors of one-dimensional subspaces invariant or inverses them; on a two-dimensional subspace it produces a rotation through the angle γ. For any orthogonal matrix A there exists an orthogonal matrix Q such that the matrix $B = Q'AQ$ is of the canonical form indicated in the problem.

Hint. Make use of problems 1567 and 1569 and apply the method of mathematical induction.

1571. $f_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$, $\quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

$f_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$,

$f_3 = \frac{1}{\sqrt{6}}(1, -2, 1)$;

1572. $f_1 = \frac{1}{2}\sqrt{2}\,(1, 1, 0)$

$f_2 = (0, 0, 1)$,

$f_3 = \frac{1}{2}\sqrt{2}\,(1, -1, 0)$; $\quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

1573. $f_1=\frac{1}{\sqrt{42+28\sqrt{2}}}(-2-\sqrt{2},\ -4-3\sqrt{2},\ \sqrt{2})$,

$f_2=\frac{1}{\sqrt{84}}(6\sqrt{2},\ -2-\sqrt{2},\ 2-\sqrt{2})$,

$f_3=\frac{1}{\sqrt{84}}(0,\ \sqrt{42-28\sqrt{2}},\ \sqrt{42+28\sqrt{2}})$;

$$B=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{-2+7\sqrt{2}}{12} & -\frac{\sqrt{42+28\sqrt{2}}}{12}\\ 0 & \frac{\sqrt{42+28\sqrt{2}}}{12} & \frac{-2+7\sqrt{2}}{12}\end{pmatrix}.$$

1574.
$$B=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & \frac{-\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix},$$

$$Q=\begin{pmatrix}\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} & 0\\ \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2}\\ \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2}\end{pmatrix}.$$

1575.
$$B=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3}\\ 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix},$$

$$Q=\begin{pmatrix}\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0\\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0\\ 0 & 0 & -1\end{pmatrix}.$$

1576.
$$B=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},$$
$$Q=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

1577.
$$B=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad Q=\frac{1}{2}\sqrt{2}\begin{pmatrix}1&0&1&0\\1&0&-1&0\\0&1&0&1\\0&-1&0&1\end{pmatrix}.$$

1578.
$$B=\begin{pmatrix}1&0&0\\0&i&0\\0&0&-i\end{pmatrix},\quad Q=\begin{pmatrix}\frac{2}{3}&\frac{2}{3}&-\frac{1}{3}i\\ -\frac{2}{3}i&\frac{1}{3}i&\frac{2}{3}\\ \frac{1}{3}i&-\frac{2}{3}i&\frac{2}{3}\end{pmatrix}.$$

1584. φ is either an identical transformation or a symmetry about subspace L of dimension $k = 0, 1, 2, \ldots, n-1$; that is, $\varphi x = x$ for any x in L and $\varphi x = -x$ for any x in the orthogonal complement L^*.

1585.
$$f_1=\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right),\quad B=\begin{pmatrix}9&0&0\\0&18&0\\0&0&-9\end{pmatrix}.$$
$$f_2=\left(\frac{2}{3},-\frac{1}{3},-\frac{2}{3}\right),$$
$$f_3=\left(\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right);$$

1586.
$$f_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\quad B=\begin{pmatrix}9&0&0\\0&9&0\\0&0&27\end{pmatrix}.$$
$$f_2=\left(\frac{1}{\sqrt{18}},-\frac{1}{\sqrt{18}},-\frac{4}{\sqrt{18}}\right),$$
$$f_3=\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right);$$

1587. $f_1=\frac{1}{\sqrt{2}}(1, -i, 0)$, $B=\begin{pmatrix}2&0&0\\0&4&0\\0&0&4\end{pmatrix}$,

$f_2=\frac{1}{\sqrt{2}}(1, i, 0)$,

$f_3=(0, 0, 1)$;

1588. $B=\begin{pmatrix}5&0\\0&-1\end{pmatrix}$; $C=\begin{pmatrix}\frac{1+i}{\sqrt{3}}&\frac{1+i}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&\frac{-2}{\sqrt{6}}\end{pmatrix}$.

1589. $B=\begin{pmatrix}2&0\\0&8\end{pmatrix}$; $C=\frac{1}{\sqrt{6}}\begin{pmatrix}2-i&1\\-1&2+i\end{pmatrix}$.

1590. *Hint.* Let E_{ij} be a matrix in which the ith row and the jth column have unity, with zero elsewhere. Consider the matrices of all transformations in an orthonormal basis:

$$E_{11}, E_{21}, \ldots, E_{n1}, E_{12}, E_{22}, \ldots, E_{n2}, \ldots, E_{nn}.$$

1591. (a) $UA = A'U$, (b) $U\overline{A} = A'U$.

1592. Two real quadratic forms can be brought to canonical form by one and the same orthogonal transformation if and only if their matrices commute. Two second order surfaces have parallel principal axes if and only if the matrices of the coefficients of the squared terms in their equations are permutable. *Hint.* Prove that the subspace L of all eigenvectors of the transformation φ that belong to one and the same eigenvalue λ is invariant under a second transformation ψ.

1593. If $A_i=\begin{pmatrix}a_{i1}\\a_{i2}\\\vdots\\a_{in}\end{pmatrix}$ and $B_i=\begin{pmatrix}b_{i1}\\b_{i2}\\\vdots\\b_{in}\end{pmatrix}$

($i = 1, 2, \ldots, n$) are columns that are orthonormal eigenvectors of P and Q respectively, then n^2 matrices $X_{ij} = A_iB_j'$ ($i, j = 1, \ldots, n$), where in the kth row and the lth column of matrix X_{ij} we have the product $a_{ik}b_{jl}$, form an orthonormal basis of eigenvectors of the transformations φ and ψ; note that any such basis is obtained in the indicated manner from certain orthonormal bases of eigenvectors of the matrices P and Q.

1594. If, for example, a linear transformation φ of a plane in an orthonormal basis is specified by the matrix $\begin{pmatrix}1&-3\\0&1\end{pmatrix}$ and the vector $x = (1, 1)$ is given by coordinates in the same basis, then $(\varphi x, x) = -1$.

1595. *Hint.* Make use of the problem 1276 (c). It is easier to consider the transformation matrices φ, ψ, χ in an orthonormal basis of eigenvectors of the transformation χ.

1596. *Hint.* The existence is proved as in problem 1276. Uniqueness is more easily proved by using the preceding problem.

1597. The eigenvalues of the transformation with matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ are equal to 3, -1, which means they are not both positive.

1598. $$\begin{pmatrix} \frac{3}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{3}{2}\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

1599. $$\begin{pmatrix} \frac{5}{2}\sqrt{2} & -\frac{3}{2}\sqrt{2} \\ -\frac{3}{2}\sqrt{2} & \frac{5}{2}\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

1600. $$\begin{pmatrix} \frac{14}{3} & \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & \frac{17}{3} & \frac{2}{3} \\ -\frac{4}{3} & \frac{2}{3} & \frac{14}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

1602. *Hint.* Find a nonsingular transformation χ such that $\chi^2 = \varphi$ and show that the transformation $\chi^{-1}\varphi\psi\chi$ is self-conjugate.

1603. *Hint.* Let φ_1 and ψ_1 be self-conjugate transformations with nonnegative eigenvalues such that $\varphi_1^2 = \varphi$ and $\psi_1^2 = \psi$. If φ is nonsingular, then, putting $\chi = \varphi_1\psi_1$, show that $\chi\chi^* = \varphi_1^{-1}\varphi\psi\varphi_1$.

1605. *Hint.* Consider the transformation matrix in a basis of eigenvectors.

1606. *Hint.* Consider an orthonormal basis in which the matrix of φ is diagonal and make a transition to the new basis.

1607. *Hint.* Show the distributivity of the transition operation from A and B to C. Consider the linear transformations φ, ψ, χ specified in some orthonormal basis of a unitary space R_n by the matrices A, B, C. Split the transformations φ and ψ into sums of nonnegative self-conjugate transformations of rank 1 with the matrices $A_1 \ldots A_r$ and $B_1 \ldots B_s$. Use problem 1606 to show that the transformation with matrix (A_i, A_j) in the same basis is nonnegative. Finally, use problem 1604 to show that the transformation χ is nonnegative.

1609. *Hint.* The proof is similar to that of the appropriate property of a self-conjugate transformation.

1610. *Solution.* The properties (a) and (b) are proved in the same way as the corresponding properties of a self-conjugate transformation.

Proof of property (c): let X and Y be the columns of coordinates of x and y respectively. Passing to coordinates in the equality $\varphi(x + yi) = \beta i(x + yi)$, we get: $AX + AYi = -\beta Y + \beta Xi$, whence,

equating real and imaginary parts, we find: $AX = -\beta Y$, $AY = \beta X$, which is proof of equality (1). Since the matrix A is real, it follows that the vector φz and the number $(\varphi z, z)$ are real for the real vector z. Therefore, $(\varphi z, z) = (z, -\varphi z) = -(z, \varphi z) = -(\varphi z, z)$ and, hence, $(\varphi z, z) = 0$. Multiplying the second of equalities (1) by y, we find $\beta(x, y) = (\varphi y, y) = 0$, whence $(x, y) = 0$. Multiplying the first of the equalities (1) by y and the second by x and then adding, we find, due to the real nature of the number $(\varphi y, x) = \beta(x, x)$, that $\beta(|x|^2 - |y|^2) = (\varphi x, y) + (\varphi y, x) = (\varphi x, y) + (x, \varphi y) = (\varphi x, y) - (\varphi x, y) = 0$, whence $|x| = |y|$.

Proof of property (d): if the transformation φ has 0 as an eigenvalue, then there is a one-dimensional invariant subspace, otherwise we pass to a unitary space. Namely, in the unitary space R'_n we take an orthonormal basis $e_1, \ldots, e_n$. The vectors in R'_n that have real coordinates in that basis form a Euclidean space R_n embedded in R'_n. In the basis $e_1, \ldots, e_n$, the transformation φ has a real skew-symmetric matrix A. In the given basis, this matrix defines the skew-symmetric transformation φ' of the unitary space R'_n, which coincides with φ on R_n. The transformation φ' has the eigenvalue $\beta_l \neq 0$. If $x + y_l$ is the corresponding eigenvector, where x and y are vectors with real coordinates, then equations (1) hold true. This means that the subspace spanned by x and y is invariant.

1611. (a) For any skew-Hermitian matrix A there exists a unitary matrix Q such that the matrix $B = Q^{-1}AQ$ is diagonal with pure imaginary elements on the diagonal, some of which may be zero.

(b) The space is a direct sum of orthogonal one-dimensional and two-dimensional subspaces that are invariant under φ. The transformation φ carries vectors of one-dimensional subspaces into zero, and on the two-dimensional subspace that corresponds to the block $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ it causes a rotation through the angle $\frac{\pi}{2}$ combined with multiplication by the number $-\beta$. For any real skew-symmetric matrix A there exists a real orthogonal matrix Q such that the matrix $B = Q^{-1}AQ$ is of the canonical form given in the text of the problem.

Hint. Make use of problems 1609 and 1610 and apply the method of mathematical induction.

1614. If A is a skew-Hermitian matrix, then the matrix $B = (E - A) \times (E + A)^{-1}$ is unitary devoid of an eigenvalue equal to -1 and, conversely, if A is a unitary matrix without an eigenvalue equal to -1, then the matrix $B = (E - A) \times (E + A)^{-1}$ is skew-Hermitian. A similar relationship obtains between real skew-symmetric and orthogonal matrices.

Solution. We consider only the case of a unitary space. In (1), let φ be a skew-symmetric transformation ($\varphi^* = -\varphi$). Then

$$\psi^* = (\varepsilon + \varphi^*)^{-1}(\varepsilon - \varphi^*) = (\varepsilon - \varphi)^{-1}(\varepsilon + \varphi) = \\ = (\varepsilon + \varphi)(\varepsilon - \varphi)^{-1} = \psi^{-1}$$

since $\varepsilon + \varphi$ and $(\varepsilon - \varphi)^{-1}$ are permutable. Hence, ψ is unitary. Note that $(\varepsilon \pm \varphi)^{-1}$ exist since the numbers ± 1 are not eigenvalues of φ [problem 1610, item (a)]. Furthermore,

$$\varepsilon + \psi = (\varepsilon + \varphi)(\varepsilon + \varphi)^{-1} + (\varepsilon - \varphi)(\varepsilon + \varphi)^{-1} = \\ = 2(\varepsilon + \varphi)^{-1} \qquad (\alpha)$$

and, hence, ψ does not have -1 as an eigenvalue. Besides, φ can be expressed in terms of ψ with the aid of an equality similar to (1). Indeed, from (α) we find $\varepsilon + \varphi = 2(\varepsilon + \psi)^{-1}$, $\varphi = 2(\varepsilon + \psi)^{-1} - \varepsilon = 2(\varepsilon + \psi)^{-1} - (\varepsilon+\psi)(\varepsilon + \psi)^{-1} = (\varepsilon - \psi)(\varepsilon + \psi)^{-1}$. Conversely, suppose in (1) φ is a unitary transformation that does not have -1 as an eigenvalue. Then $\psi^* = (\varepsilon + \varphi^*)^{-1}(\varepsilon - \varphi^*) = (\varepsilon + \varphi^{-1})^{-1}(\varepsilon - \varphi^{-1}) = (\varphi + \varepsilon)^{-1}\varphi\varphi^{-1}(\varphi - \varepsilon) = -(\varepsilon - \varphi) \times (\varepsilon + \varphi)^{-1} = -\psi$ since $\varepsilon - \varphi$ and $(\varepsilon + \varphi)^{-1}$ are permutable. Hence, ψ is a skew-symmetric transformation. Besides, φ can be expressed in terms of ψ with the aid of an equality similar to (1). This is again proved with the aid of (α) just exactly as done above. Thus, (1) maps all skew-symmetric transformations into unitary transformations that do not have the number -1 as an eigenvalue, and conversely; note that one of these mappings is the inverse of the other. This proves that both mappings are one-to-one.

1616. If the matrix A is skew-Hermitian (or real skew-symmetric), then the matrix e^A is unitary (respectively orthogonal).

1617. *Solution.* By problem 1559, the transformation e^{φ} is self-conjugate. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of φ, then they are real and, by problem 1161, the eigenvalues of e^{φ} are $e^{\lambda_1}, \ldots, e^{\lambda_n}$, that is, e^{φ} is positive definite. Let us now show that to distinct self-conjugate transformations φ and φ' there correspond distinct transformations e^{φ} and $e^{\varphi'}$. Let $e^{\varphi} = e^{\varphi'}$; φ has an orthonormal basis of eigenvectors $a_1, \ldots, a_n$, where $\varphi a_i = \lambda_i a_i$ $(i = 1, 2, \ldots, n)$. Let a' be any eigenvector of the transformation φ' with the eigenvalue λ'; $a' = \sum_{i=1}^{n} x_i a_i$. Then, by problem 1464, $e^{\varphi'}a' = e^{\lambda'}a' = \sum_{i=1}^{n} e^{\lambda'} x_i a_i$. On the other hand, $e^{\varphi}a' = \sum_{i=1}^{n} x_i e^{\varphi} a_i = \sum_{i=1}^{n} x_i e^{\lambda_i} a_i$; since $e^{\varphi'}a' = e^{\varphi}a'$, it follows that $x_i = 0$ for all those i for which $e^{\lambda_i} \neq e^{\lambda'}$, and $e^{\lambda_i} = e^{\lambda'}$ if $x_i \neq 0$. Since λ_i and λ' are real, it follows from $x_i \neq 0$ that $\lambda_i = \lambda'$. Therefore, $\varphi a' = \sum_{i=1}^{n} x_i \varphi a_i = \sum_{i=1}^{n} x_i \lambda_i a_i = \sum_{i=1}^{n} x_i \lambda' a_i = \lambda' a' = \varphi' a'$. Since φ' has a basis of eigenvectors and coincides, on this basis, with φ, it follows that $\varphi = \varphi'$. Let ψ be any positive definite transformation. There exists an orthonormal basis in which the matrix ψ is diagonal with positive elements $\mu_1, \mu_2, \ldots, \mu_n$ on the diagonal. Suppose $\lambda_i = \ln \mu_i$ $(i = 1, 2, \ldots, n)$, where λ_i is the real value of the logarithm, and let the transformation φ in the same basis be given by a diagonal matrix with elements $\lambda_1, \ldots, \lambda_n$ on the diagonal. The transformation φ is self-conjugate and $\psi = e^{\varphi}$.

1625. *Hint.* Apply problem 1507.

1626. If r is the rank of φ, then the number of such transformations is equal to k^r.

1627. *Hint.* Prove the equality

$$(\varphi x - \lambda x,\ \varphi x - \lambda x) = (\varphi^* x - \bar{\lambda} x,\ \varphi^* x - \bar{\lambda} x).$$

1628. *Hint.* Use the preceding problem.

1629. *Hint.* Use problem 1627.

1630. *Hint.* Make use of problems 1627 and 1629.

1631. *Hint.* Apply problem 1629 several times.

1632. *Hint.* To prove necessity, use the two preceding problems. To prove sufficiency, show that the transformation φ of unitary space, which transformation has the normal property, has an orthonormal basis of eigenvectors. Reduce the case of Euclidean space to that of unitary space.

1633. *Hint.* Use the preceding problem.

Supplement

1634. (1) Yes, (2) yes, (3) yes, (4) yes, (5) no, (6) no, (7) no, (8) yes, (9) no, (10) yes, (11) yes, (12) no, (13) yes, (14) yes, (15) yes, (16) yes, (17) no, (18) yes, (19) no, (20) yes, (21) yes, (22) yes, (23) yes, (24) yes, (25) no, (26) yes, (27) yes, (28) yes, (29) yes, (30) yes, (31) yes, (32) no, (33) no, (34) yes, (35) no, (36) yes.

1637. *Hint.* First method: show that $|a| = 1$ for any a of the given group G of order n. For $n > 1$ take, in G, an element $b = \cos\psi + i\sin\psi$ with least positive argument ψ and show that

$$G = \{1, b, b^2, \ldots, b^{n-1}\}.$$

Second method: using the Lagrange theorem, show that $a^n = 1$ for any a in G.

1638. (a) One group—a cyclic group of the third order—with elements e, a, b and the array

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

In the representation as substitutions, we can set e for the unit element, $a = (1\ 2\ 3)$, $b = (1\ 3\ 2)$.

(b) Two groups: (1) a cyclic group of the fourth order with elements e, a, b, c and the array

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

In the representation as substitutions we can set e for the unit element, $a = (1\ 2\ 3\ 4)$, $b = (1\ 3)\ (2\ 4)$, $c = (1\ 4\ 3\ 2)$;

(2) the four-group with elements e, a, b, c and array

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

In the representation as substitutions we can set e for the unit element, $a = (1\ 2)\ (3\ 4)$, $b = (1\ 3)\ (2\ 4)$, $c = (1\ 4)\ (2\ 3)$.

(c) Two groups: (1) a cyclic group of sixth order with elements e, a, b, c, d, f and the array

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	c	d	f	e
b	b	c	d	f	e	a
c	c	d	f	e	a	b
d	d	f	e	a	b	c
f	f	e	a	b	c	d

In the representation as substitutions we can set e for the unit element, $a = (1\ 2\ 3\ 4\ 5\ 6)$, $b = (1\ 3\ 5)\ (2\ 4\ 6)$, $c = (1\ 4)\ (2\ 5)\ (3\ 6)$, $d = (1\ 5\ 3)\ (2\ 6\ 4)$, $f = (1\ 6\ 5\ 4\ 3\ 2)$;

(2) a symmetric group of the third degree with elements e, a, b, c, d, f and the array

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e

In the representation as substitutions we can set e for the unit element, $a = (1\ 2\ 3)$, $b = (1\ 3\ 2)$, $c = (1\ 2)$, $d = (2\ 3)$, $f = (1\ 3)$.

Hint. Show that if in a group G of order n there is a set H of k elements, $k < n$, which is itself a group under the operation of multiplication specified in G, then by multiplying all elements in H by the element x not in H, we obtain k new elements of G. For that reason $k \leqslant \frac{n}{2}$. For H we can take the set of elements e, a, a^2, . . ., a^{k-1}, where $a^k = e$. For example, in the case (2) of (c), that is, for a noncyclic group G of sixth order, it must be true that $k \leqslant 3$. If it were true that $a^2 = e$ for any a in G, then the four elements e, a, b, ab

would form a group, but that is impossible. Hence there is an element a for which $a^2 = b \neq e$ but $a^3 = e$. Multiplying the elements e, a, a^2 by the new element c, we obtain all six elements of the group G in the form e, a, $a^2 = b$, c, $ac = d$, $a^2c = f$. It must be shown that $c^2 = d^2 = f^2 = e$ and $ca = a^2c = f$. For example, if it were true that $ca = ac$, then by multiplying on the left first by c and then by a^2, we would have $a^2cac = fd = e$, whence $fd = d^2$ and $f = d$, which is impossible.

1639. The tetrahedron group is of order 12, the cube and octahedron of order 24, the dodecahedron and icosahedron of order 60. *Hint.* Consider rotations that carry the given vertex A into some vertex B (not necessarily different from A) and show that the order of the group is equal to nk, where n is the number of vertices and k is the number of edges emanating from one vertex.

1643. *Hint.* With every element x of the given group G associate a mapping $a \to ax$ for every element a of G.

1646. ± 1.

1648. *Hint.* (a) Consider $(ab)^{p^r}$ and $(ab)^{p^s}$, where p is the order of ab. (b) Consider $(ab)^p$, where p is the order of ab, and show that $a^p = b^{-p} = e$.

Example 1. Concerning the elements $a \neq e$, $b = a^{-1}$, condition (1) is fulfilled but condition (2) is not. The assertion (b) is not fulfilled since the orders of a and b are equal to each other but are not equal to unity, while the order of $ab = e$ is equal to unity.

Example 2. The elements $a = (1\ 2)$, $b = (1\ 2\ 3)$ of the symmetric group S_3 have relatively prime orders 2 and 3. The condition (1) is not fulfilled since $ab = (1\ 3)$, $ba = (2\ 3)$ but condition (2) is fulfilled. Assertion (b) is not fulfilled: a is of order 2, b is of order 3, and ab is of order 2.

(c) Raise both sides of $a^k = b^l$ to the power s, which is equal to the order of b.

(d) *Example 3.* In the cyclic group $\{a\}$ of eighth order the elements a, a^3, a^5 are of order 8, but $aa^3 = a^4$ is of order 2, $aa^5 = a^6$ is of order 4.

1649. (2) and (3) of (1); (4) and (11) of (10); (1), (2), (3), (13), (14) of (8); (15) of (16); (20) and (21) of (18); (20) of (21); (24) of (23); (23) and (24) of (26); (29) and (30) of (31); (34) of (36).

1653. An infinite cyclic group, all cyclic groups of prime orders and the unit group.

1654. (a) $G = \{a\}, \{a^2\}, \{a^3\}, \{e\}$;

(b) $G = \{a\}, \{a^2\}, \{a^3\}, \{a^4\}, \{a^6\}, \{a^8\}, \{a^{12}\}, \{e\}$;

(c) $G = \{e, a, b, c\}, \{a\}, \{b\}, \{c\}, \{e\}$;

(d) using the substitution notation in the cycles, we obtain the subgroups

S_3, $\{(1\ 2\ 3)\}$, $\{(1\ 2)\}$, $\{(1\ 3)\}$, $\{(2\ 3)\}$, $\{e\}$;

(e) the normal divisors are S_3, $\{(1\ 2\ 3)\}$, $\{e\}$.

(f) *Hint.* The decomposition into cycles of the substitution of A_4 can only contain cycles of length 1, two cycles of length 2 or one cycle of length 3. Therefore A_4 does not have a cyclic subgroup of the sixth order [see problem 1648 (a) and (b)] and all its elements of the second order are permutable. Hence, A_4 does not have a subgroup isomorphic to S_3. But any group of the sixth order is either cyclic or isomorphic to S_3 [problem 1638 (c)].

1655. In G we choose any elements: first $a \neq e$, then $b \neq e, a$, and then $c \neq e, a, b, ab$. Then the remaining elements of group G are ab, ac, bc, abc. G is an Abelian group (problem 1636). The group G has the following 16 subgroups: $\{e\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, ab\}$, $\{e, ac\}$, $\{e, bc\}$, $\{e, abc\}$, $\{e, a, b, ab\}$, $\{e, a, c, ac\}$, $\{e, b, c, bc\}$, $\{e, a, bc, abc\}$, $\{e, b, ac, abc\}$, $\{e, c, ab, abc\}$, $\{e, ab, ac, bc\}$, $\{e, a, b, c, ab, ac, bc, abc\} = G$.

1657. Written additively, all subgroups are of the form

$$G_0 = \{a\},\ G_1 = \{pa\},\ G_2 = \{p^2a\},\ \ldots,\ G_{k-1} = \{p^{k-1}a\},\ G_k = \{p^ka\} = \{0\}.$$

They form a decreasing chain of subgroups of the orders, respectively, $p^k, p^{k-1}, p^{k-2}, \ldots, p, 1$.

Hint. Use problem 1656 (b) or show that the subgroup $\{sa\}$, where $0 < s < p^k$ coincides with the subgroup $\{p^la\}$, where $s = p^lt$, $0 \leqslant l < k$ and t is not divisible by p.

1658. (a) *Hint.* Decompose the substitution into cycles and verify that $(i_1i_2i_3 \ldots i_k) = (i_1i_2)(i_1i_3) \ldots (i_1i_k)$.

(b) *Hint.* Verify the equality $(1i)(1j)(1i) = (ij)$.

(c) *Hint.* Verify that the product of two transpositions can be expressed in the following manner in terms of triple cycles:

$$(ij)(ik) = (ijk),\ (ij)(kl) = (ijk)(ilk).$$

(d) *Solution.* Let G be a subgroup of the alternating group A_n generated by the set of the indicated triple cycles, and let i, j, k be distinct numbers exceeding two (for $n = 3$ the assertion is obvious and for $n = 4$ the proof given below is reduced). Together with the cycle $(1\ 2\ i)$ the group G contains the inverse element $(i\ 2\ 1)$, and then G contains

$$(1\ 2\ j)(1\ 2\ i)(j\ 2\ 1) = (1\ i\ j);\quad (j\ 2\ 1)(i\ 2\ 1)\ (1\ 2\ j) = (2\ i\ j)$$

For $n = 4$ the group G contains all triple cycles. For $n > 4$ it contains $(1\ 2\ k)(1\ i\ j)(k\ 2\ 1) = (i\ j\ k)$. Hence, G contains all triple cycles and, by item (c), coincides with A_n.

1660. *Hint.* Let K be the set of all elements of a group G, which elements do not belong to H, and let a be any element in K. Show that by multiplying a by all elements taken from H, we obtain all elements of K. From this fact derive that by multiplying a by all elements taken from K, we obtain all elements of H. In particular, a^2 belongs to H.

1661. An example is the four-group with elements e, a, b, c (see answer to problem 1638). It has three cyclic subgroups of the second order: $\{a\}$, $\{b\}$ and $\{c\}$.

Hint. Prove that by squaring all triple cycles we again obtain all triple cycles and make use of problems 1658 and 1660.

1662. (a) *Hint.* To every rotation of the tetrahedron $ABCD$ there corresponds a substitution of its vertices. To a product of two rotations there corresponds a product of the respective substitutions. To two distinct rotations s and t there correspond two distinct substitutions, for otherwise the nonidentical rotation st^{-1} would be associated with the identical substitution that maintains all vertices fixed.

From the answer to problem 1639, the tetrahedron group is isomorphic to a subgroup of twelfth order of the symmetric group S_4. Now, one can either verify that all substitutions corresponding to rotations of the tetrahedron are even or one may take advantage of problem 1661.

(b) *Solution.* The centres of the faces of the octahedron are vertices of a cube. Therefore the groups of the cube and the octahedron are isomorphic. To each rotation of the cube there corresponds a substitution of its four diagonals. A product of rotations is associated with a product of the respective substitutions. Let us consider all rotations of the cube. These are: the identical rotation, eight rotations about the diagonals through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, six rotations about the axes passing through the midpoints of the opposite edges through the angle π, and nine rotations about the axes, passing through the centres of the opposite faces, through the angles $\frac{\pi}{4}$, $\frac{2\pi}{4}$ and $\frac{3\pi}{4}$. The number of these rotations comes to $1 + 8 + 6 + 9 = 24$. According to the answer to problem 1639, these exhaust all rotations of the cube. A direct verification shows that only under the identical rotation do all four diagonals remain fixed. From this fact, as in item (a), we derive that the cube group is isomorphic to the group of substitutions of four elements (it has order 24), that is, to the symmetric group S_4.

(c) *Solution.* The centres of the faces of the dodecahedron are vertices of an icosahedron. Therefore the dodecahedron and icosahedron groups are isomorphic. For each edge of the icosahedron there is one opposite parallel edge and two pairs of edges perpendicular to it: the edges of one pair begin at the vertices of the faces adjacent to the given edge, and the edges of the other pair belong to the faces having as vertices the endpoints of the given edge. The edges of one of the pairs are parallel, the edges of distinct pairs are perpendicular to each other. Thus, all 30 edges break down into five systems with six in each system. The edges of one system are either parallel or perpendicular, those of different systems are not parallel and not perpendicular. With each system of edges there is associated an octahedron whose vertices are the midpoints of the edges of the given system. Thus are defined five octahedrons inscribed in the icosahedron. To each rotation of the icosahedron there corresponds a substitution of the five indicated systems of edges (or of the octahedrons that correspond to them). A product of two rotations is associated with the product of the corresponding substitutions. Let us consider all rotations of the icosahedron. They include the identical rotation; 24 rotations about each of the six axes passing through opposite vertices, through the angles $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $\frac{6\pi}{5}$ and $\frac{8\pi}{5}$, 20 rotations about each of the ten axes that pass through the centres of opposite faces, through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, 15 rotations about each of the fifteen axes passing through the midpoints of the opposite edges, through the angle π. The total number of rotations is: $1 + 24 + 20 + 15 = 60$. According to the answer to problem 1639 they exhaust all rotations of the icosahedron. A direct verification shows that for each nonidentical rotation there is an edge that is carried by the given rotation into another edge that is neither parallel nor perpendicular to the given edge. For this reason,

an identical substitution of the systems of edges corresponds only to an identical rotation. From this fact, as in item (a), we derive that the icosahedron group is isomorphic to the subgroup of order 60 of the symmetric group S_5. According to problem 1661, this subgroup coincides with the alternating group A_5.

1668. *Hint.* (a) Apply problem 1667. (b) Show that each coset contains exactly one substitution that leaves the number 4 fixed in place.

1669. If in a decomposition of the given substitution s into independent cycles there are k_i cycles of length l_i, $i = 1, 2, \ldots, r$ with all cycles (even cycles of length 1) included, then the number of substitutions that commute with the substitution s is equal to $\prod_{i=1}^{r} (k_i)! \cdot l_i^{k_i} i$. Assuming $0! = 1$, we can write the desired number differently. Let j_i be the number of cycles of length i that enter into the decomposition of the substitution s, where $i = 1, 2, \ldots, n$, and if there are no cycles of length i in the decomposition, then j_i is put equal to 0. Then the desired number is equal to $\prod_{i=1}^{n} (j_i)! \cdot i^{j_i} i$.

Hint. Cycles of one and the same length l that appear in the decomposition s can, under a transformation via the substitution x that commutes with s, only be interchanged; here the first number of any cycle can pass into any number of any cycle of the same length that enters into the decomposition of the substitution s.

1670. *Hint.* Consider the commutator $h_1 h_2 h_1^{-1} h_2^{-1}$ of these elements.

1673. *Hint.* Make use of the following problems: 1671 in the case (a), 1672 in the case (b).

1674. If in the decomposition of the given substitution s into independent cycles there occur k_i cycles of length l_i, $i = 1, 2, \ldots, r$ with all cycles (including cycles of length 1) taken into account, then the desired number is equal to $\dfrac{n!}{\prod_{i=1}^{r} (k_i)! \cdot l_i^{k_i} i}$. This number may be written differently if we make use of a different expression of the denominator—that given in the answer to problem 1669.

1675. *Hint.* Denote by a_k the number of classes of conjugate elements with p^k elements and, using problem 1673, show that $a_0 + a_1 p + a_2 p^2 + \ldots = p^n$.

1676. *Hint.* If H contains the cycle $(\alpha\ \beta\ \gamma)$ and α', β', γ' are arbitrary distinct numbers from 1 to n, then transform the cycle $(\alpha\ \beta\ \gamma)$ via the substitution

$$x = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \ldots \\ \alpha' & \beta' & \gamma' & \delta' & \varepsilon' & \ldots \end{pmatrix},$$

where δ' and ε' are chosen so that the substitution x is even. Make use of the problems 1667 and 1658 (c).

1677. *Solution.* (a) All 60 rotations that make up the icosahedron group are given in the answer to problem 1662 (c). The identical rotation is the unit of the group and constitutes one class. The conjugate elements have the same order. The elements of the fifth order are 24

rotations through the angles $\frac{2k\pi}{5}$, $k = 1, 2, 3, 4$ about each of the six axes passing through opposite vertices. By a rotation about the vertex A through the angle α we will mean a rotation about the axis passing through A and the opposite vertex, through the angle α counterclockwise when looking along the axis from A to the opposite vertex. At each vertex we note one of the plane angles with the given vertex. Each rotation of the icosahedron is fully characterized by indicating the vertex B into which the given vertex A passes (B may coincide with A), and the plane angle at B, into which the marked angle at A passes. Therefore each rotation x that carries A into B is represented in the form of a product $x = yz$, where y carries the marked angle A into the marked angle B, and z is a rotation about the vertex B through the angle α. The inverse element $x^{-1} = z^{-1}y^{-1}$ is a product of the rotation z^{-1} about B through the angle $-\alpha$ and the rotation y^{-1} that carries the marked angle B into the marked angle A. Now suppose g is the rotation about the vertex A through the angle α and x is any element of the group, which element carries A into B. Representing x in the form of a product, $x = yz$, as indicated above, we find that the conjugate element $x^{-1}gx = z^{-1}y^{-1}gyz$ is again a rotation through the angle α but this time about vertex B. In particular, if A and B are opposite vertices, then the rotation about B through the angle α coincides with the rotation about A through the angle $2\pi - \alpha$. Thus, all rotations about the vertices through the angles $\frac{2\pi}{5}$ and $\frac{8\pi}{5}$ belong to one class of conjugate elements, just as all the rotations through the angles $\frac{4\pi}{5}$ and $\frac{6\pi}{5}$. We now show that the rotations g_1 and g_2 about the vertex A through the angles $\frac{2\pi}{5}$ and $\frac{4\pi}{5}$ belong to distinct classes. If x carries A into a different vertex B, then $x^{-1}g_1x$ is a rotation about B and either will not be a rotation about A or (if B is opposite to A) will be a rotation about A through the angle $\frac{8\pi}{5}$, that is, $x^{-1}g_1x \neq g_2$. But if x is a rotation about A, then g_1 and x are elements of a cyclic (and, hence, commutative) subgroup of rotations about A and again $x^{-1}g_1x = g_1 \neq g_2$.

Thus, all elements of the fifth order split up into two classes with 12 elements in each class. Similarly, marking one plane angle of each face and one vertex of each edge, we see that 20 elements of the third order (rotations through the angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axes passing through the centres of opposite faces) constitute one class and 15 elements of the second order (rotations through the angle π about the axes passing through the midpoints of opposite edges) also constitute one class.

(b) The normal divisor should consist of integral classes, should contain the unit element, and its order should divide the order 60 of the icosahedron group. According to item (a), classes of conjugate elements contain 1, 12, 12, 20, and 15 elements respectively. Using these numbers, it is only possible to form two sums such that include the summand 1 and divide the number 60, namely, 1 and 60. This

yields only two normal divisors: the unit subgroup and the entire group.

1678. *Hint.* Use the problems 1662 (c) and 1677 (b).

1681. A homomorphism is fully determined by the image of the generating element a. Below are indicated the possible images of that element:

(a) any element of the group; the number of homomorphisms is equal to n;

(b) e, b^3, b^6, b^9, b^{12}, b^{15};

(c) e, b, b^2, b^3, b^4, b^5;

(d) e, b^5, b^{10}; (e) e.

1683. (a) A cyclic group $\{\varphi\}$ of the fourth order, where $a\varphi = a^2$;
(b) a cyclic group $\{\varphi\}$ of the second order, where $a\varphi = a^5$;
(c) a field of residues modulo 5;
(f) a ring of residues modulo 6;
(g) a ring of residues modulo n.

1685. (a) A cyclic group of order n;
(b) a cyclic group of order 5;
(c) a cyclic group of order 6;
(d) a cyclic group of order 2.

1688. *Hints.* In the cases (d), (e), and (h), consider the mapping $f(z) = z^n$, and in the case (f) consider the mapping $g(z) = \frac{z^n}{|z|^n}$.

1691. The subgroup $\{(1\ 2)\}$ in group S_3 has index 3 but does not contain the element $(1\ 3)$ of order 2.

1693. *Hint.* Assuming that G/Z is a cyclic group, choose element a in the class that serves as the generating element of that group and show that a and Z generate the entire group G.

1694. *Solution.* Use induction on the order n of the group G. For $n = 2$ the group G is a cyclic group of the second order, and the theorem holds with respect to it. Let the theorem hold true for all groups whose orders are less than n and let G be a group of order n. First let G be commutative. Take any element a that differs from the unit element e of G. Its order is $k > 1$. If k is divisible by p, $k = pq$, then the element a^q is of order p. If k is not divisible by p, then the order n' of the factor-group $G' = G/\{a\}$ of G with respect to the cyclic subgroup $\{a\}$ is equal to $\frac{n}{k} < n$ and is divisible by p. By the hypothesis of our induction, G' contains the element b' of order p. Let b be an element of G that appears in the coset b'. From $b'^p = e'$, where e' is the unit element of G', it follows that b^p is contained in the subgroup $\{a\}$, that is, $b^p = a^l$, whence $b^{pk} = a^{lk} = e$. If $b^k = e$, then $b'^k = e'$ and k is divisible by the order p of element b', which is impossible. Hence, $b^{kp} = e$ but $b^k \neq e$, i.e. the element b^k is of order p.

Now let the group G be noncommutative. If there is a subgroup H different from G whose index is not divisible by p, then the order of H is less than n and is divisible by p. By the induction hypothesis, H contains an element of order p. But if the indices of all subgroups of group G different from G are divisible by p, then the number of elements conjugate to any element of G not appearing in its centre Z

(problem 1664) is divisible by p (problem 1671). Since the order n of the group G is also divisible by p, it follows that the order of the centre Z is divisble by p and is less than n, since G is noncommutative. By the induction hypothesis, Z contains an element of order p.

1695. *Hint.* Make use of the preceding problem.

1701. (a) $\{a\} = \{3a\} + \{2a\}$;
(b) $\{a\} = \{4a\} + \{3a\}$;
(c) $\{a\} = \{15a\} + \{20a\} + \{12a\}$;
(d) $\{a\} = \{225a\} + \{100a\} + \{36a\}$.

1702. *Hint.* In the case of (c) take advantage of problem 1700 (b).

1703. *Hints.* (a) For A and B take, respectively, the sets of all elements a and b of G for which $pa = 0$ and $qb = 0$; (b) consider the decompositions $n = p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$ of order n of group G into prime factors, and apply (a).

1704. (a) $G(3)$; (b) $G(4)$; $G(2, 2)$; (c) $G(2, 3)$; (d) $G(8)$, $G(2, 4)$, $G(2, 2, 2)$; (e) $G(9)$, $G(3, 3)$; (f) $G(4, 3)$, $G(2, 2, 3)$; (g) $G(16)$, $G(2, 8)$, $G(4, 4)$, $G(2, 2, 4)$, $G(2, 2, 2, 2)$; (h) $G(8, 3)$, $G(2, 4, 3)$, $G(2, 2, 2, 3)$; (i) $G(2, 3, 5)$; (j) $G(4, 9)$, $G(2, 2, 9)$, $G(4, 3, 3)$, $G(2, 2, 3, 3)$; (k) $G(16, 3)$, $G(2, 8, 3)$, $G(4, 4, 3)$, $G(2, 2, 4, 3)$, $G(2, 2, 2, 2, 3)$; (l) $G(4, 3, 5)$, $G(2, 2, 3, 5)$; (m) $G(9, 7)$, $G(3, 3, 7)$; (n) $G(8, 9)$, $G(2, 4, 9)$, $G(2, 2, 2, 9)$, $G(8, 3, 3)$, $G(2, 4, 3, 3)$, $G(2, 2, 2, 3, 3)$; (o) $G(4, 25)$, $G(2, 2, 25)$, $G(4, 5, 5)$, $G(2, 2, 5, 5)$.

1705. If Z_k is a cyclic group of order k and Z is an infinite cyclic group, then the desired direct decomposition of the factor-group G/H has the form:

(a) $Z_2 + Z_2 + Z_3$; (b) $Z_3 + Z_4$; (c) $Z_2 + Z_3 + Z_3$; (d) $Z_2 + Z_4$; (e) $Z_4 + Z$; (f) $Z_2 + Z_2 + Z$; (g) Z_3; (h) $Z + Z$; (i) Z; (k) G/H is the zero group; the desired decomposition does not exist.

1706. (c) The group G can be uniquely decomposed into a direct sum of subgroups: $G = A_1 + A_2 + \ldots + A_s$, where A_i is a cyclic subgroup of order p_i. Any nonzero subgroup H of group G is a direct sum of certain of the subgroups A_i. The number of all subgroups is equal to 2^s. *Hint.* Use the item (b) and show that if h is the generator of the subgroup H, then H is a direct sum of those subgroups A_i which contain the nonzero components of the element h.

1707. *Hints.* (c) To prove the decomposition $G = H + K$ take any element a_1 outside H, then any element a_2 outside $\{H, a_1\}$ and so forth, and put $K = \{a_1, a_2, \ldots\}$.

(d) Any subgroup H of order p^l can be decomposed into a direct sum l of cyclic subgroups of order p. Suppose the decomposition is of the form

$$H = \{a_1\} + \{a_2\} + \ldots + \{a_l\}.$$

We find the number of all systems $(a_1, a_2, \ldots, a_l)$ defined in the appropriate manner for all subgroups H of order p^l. Since $a_1 \neq 0$, it follows that for a_1 we have $p^k - 1$ possibilities. Since a_2 lies outside the cyclic subgroup $\{a_1\}$, it follows that for a_2 we have $p^k - p$ possibilities, and so forth. In similar fashion we find the number of all systems $(a_1, a_2, \ldots, a_l)$ that yield one group H of order p^l. The number of all subgroups of order p^l is equal to the quotient of the two numbers that were found.

1708. *Hint.* First consider the case of the primary group, then take the decomposition of the group into primary components [problem 1703 (b)] and apply problem 1700 (b).

1709. A ring. **1710.** A ring.

1711. A ring. For $n = 0$ we obtain a zero ring consisting of the single number 0, which is the unit element of the ring and is at the same time its inverse element. The zero ring is not a field since a field must contain more than one element.

1712. A field. **1713.** A field. **1714.** A field.

1715. A ring. **1716.** A field. **1717.** A ring.

1718. A field. **1719.** A ring. **1720.** A ring.

1721. A ring. **1722.** A ring. **1723.** A ring.

1724. Matrices with rational a, b form a field, those with real a, b form a ring but not a field.

1725. Polynomials in sines and cosines and polynomials in cosines alone form a ring, but polynomials in sines alone do not form a ring. *Hint.* To prove that polynomials in sines do not form a ring, make use of the fact that the product of two odd functions is an even function.

1726. They do not. *Hint.* Use the irreducibility of the polynomials $x^3 - 2$ over the field of rational numbers to prove that $\sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{4}$ does not belong to the set under consideration.

1727. $\frac{1}{43}(5 + 9\sqrt[3]{2} - \sqrt[3]{4})$. *Hint.* To prove uniqueness, take advantage of the irreducibility of the polynomial $x^3 - 2$ over the field of rational numbers. Use the method of undetermined coefficients to seek the inverse element.

1728. $x^{-1} = \frac{1}{208}(19 - \sqrt[3]{5} + 11\sqrt[3]{25})$.

1729. *Hint.* Use the property of an irreducible polynomial of being relatively prime to any polynomial of lower degree.

1730. $\beta^{-1} = \frac{1}{405}(101 + 37\alpha + 4\alpha^2)$.

Hint. If $\varphi(x) = x^2 - x + 3$, then use the method of undetermined coefficients to find the polynomials $f_1(x)$ of first degree and $\varphi_1(x)$ of second degree that satisfy the equation $f(x) f_1(x) + \varphi(x) \varphi_1(x) = 1$, and set $x = \alpha$ in that equation.

1732. For example, $f_1(x) = \begin{cases} 0 \text{ for } x \leqslant 0, \\ x \text{ for } x \geqslant 0, \end{cases}$

$$f_2(x) = \begin{cases} x \text{ for } x \leqslant 0, \\ 0 \text{ for } x \geqslant 0. \end{cases}$$

1734. The zero divisors are of the form $(a, 0)$, where $a \neq 0$, and $(0, b)$, where $b \neq 0$.

1737. The matrices in which the element in the upper left-hand corner is nonzero will not be left divisors of zero but will be right divisors of zero.

1738. *Hint.* Remove the brackets in the product $(a + b)(e + e)$ in two different ways.

1740. Matrices of order $n \geqslant 2$ with elements taken from the given field form a ring with several left units, provided that all rows beginning with the second consist of zeros; and under a similar condition for columns, they form a ring with several right units.

1742. *Hint.* Let a be a nonzero element of a ring. Show that the correspondence $x \to ax$, where x is any element, is a one-to-one mapping of the given ring onto itself.

1743. *Hint.* Make use of problem 1742.

1747. *Hint.* Find the matrices E, I, J, K that correspond to the unit elements 1, i, j, k and verify the multiplication table for them: $I^2 = J^2 = K^2 = -E$, $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$.

1749. Two such automorphisms are possible; the identical automorphism and the automorphism that carries each number into its conjugate.

1750. *Hint.* Show that any number field contains the number 1, the integers, and, finally, fractions.

1751. *Hint.* Consider the images of unity, the integers, and of fractions.

1752. *Hint.* Show that a positive number, being the square of a real number, passes into a positive number. Then, taking advantage of the fact that there is a rational number between two distinct real numbers and also using the preservation of rational numbers, prove the invariability of any real number.

1753. Only two such mappings are possible: the identical mapping and a mapping that carries any complex number into its conjugate.

1756. The system is inconsistent modulo 3, but the system modulo 5 has a unique solution: $x = 2$, $y = 3$, $z = 2$.

1757. The system modulo 5 is inconsistent, but the system modulo 7 has a unique solution: $x = 2$, $y = 6$, $z = 5$.

1758. (a) $x + 2$, (b) 1. 1759. (a) 1, (b) $5x + 1$.

1760. (a) $x^2 + x + 2$, (b) 1.

1761. (a) *Solution.* Suppose that $f(x)$ and $g(x)$ have a common divisor $d(x)$ of positive degree over the field of rational numbers. Then $f(x) = a(x)\, d(x)$, $g(x) = b(x)\, d(x)$, where $a(x)$, $b(x)$, $d(x)$ are polynomials with rational coefficients. Taking out the common denominators and the largest common divisors of the numerators of the coefficients and applying the Gaussian lemma on the product of primitive polynomials, we obtain: $f(x) = a_1(x)\, d_1(x)$, $g(x) = b_1(x) \times \times d_1(x)$, where all polynomials have integral coefficients, the degree of $d_1(x)$ is equal to the degree of $d(x)$ and the leading coefficient of $d_1(x)$ is not divisible by p. Passing to the field of residues modulo p, we obtain the largest divisor of positive degree for $f(x)$ and $g(x)$ over that field, but that is impossible.

(b) The polynomials $f(x) = x$, $g(x) = x + p$ are relatively prime over the field of rational numbers and are equal to x, that is, they are not relatively prime over the field of residues modulo p.

1762. *Hint.* If $f(x)$ and $g(x)$ are relatively prime, then, by obtaining the equality $f(x)\, u(x) + g(x)\, v(x) = c$ where $u(x)$, $v(x)$ are polynomials with integral coefficients and c is an integer, prove that $f(x)$ and $g(x)$ are relatively prime over the field of residues modulo any prime p that does not divide c.

When proving the converse, make use of problem 1761.

1763. $(x + 1)^3 (x^2 + x + 1)$. 1764. $(x + 3)(x^2 + 4x + 2)$.

1765. $(x^2+1)(x^2+x+2)$. 1766. $(x^2+x+1)(x^2+2x+4)$.
1767. $f_1 = x^2$, $f_2 = x^2+1 = (x+1)^2$, $f_3 = x^2+x = x(x+1)$, $f_4 = x^2+x+1$ is irreducible.
1768. $f_1 = x^3$, $f_2 = x^3+1 = (x+1)(x^2+x+1)$, $f_3 = x^3+x = x(x+1)^2$, $f_4 = x^3+x^2 = x^2(x+1)$, $f_5 = x^3+x+1$ is irreducible, $f_6 = x^3+x^2+1$ is irreducible.
$f_7 = x^3+x^2+x = x(x^2+x+1)$, $f_8 = x^3+x^2+x+1 = (x+1)^3$.

1769. $f_1 = x^2+1$, $f_2 = x^2+x+2$, $f_3 = x^2+2x+2$.

1770. $f_1 = x^3+2x+1$, $f_2 = x^3+2x+2$,
$f_3 = x^3+x^2+2$, $f_4 = x^3+2x^2+1$,
$f_5 = x^3+x^2+x+2$, $f_6 = x^3+x^2+2x+1$,
$f_7 = x^3+2x^2+x+1$, $f_8 = x^3+2x^2+2x+2$.

1771. *Hint.* Make use of the Gaussian lemma, and from the factorization of $f(x)$ into two factors with rational coefficients obtain a factorization into two factors with integral coefficients. The polynomial $f(x) = px^2+(p+1)x+1 = (px+1)(x+1)$ is reducible over the field of rational numbers, but with respect to modulus p it is equal to $x+1$ and, hence, is irreducible.
1772. *Hint.* Assume the contrary, apply a decomposition of the group G into a direct product of primary cyclic subgroups, make use of problem 1700 (c) and the theorem that the equation $x^n = 1$ does not have more than n distinct roots in the field Z_p.
1773. *Solution.* First prove the lemma from group theory. If two elements a and b of a cyclic group G are not squares, then their product is a square.
The set H of elements taken from G that are squares is a subgroup. The factor-group G/H is cyclic. If $C = cH$ is its generator, then from $c^2 \in H$ follows $C^2 = c^2H = H$. Hence, either $H = G$ or G/H is a group of the second order and $ab \in aH \cdot bH = H$, that is, ab is a square.
From this it follows that one of the numbers 2, 3, 6 is congruent to a square with respect to any prime number modulus p. Indeed, for $p = 2$ we have $2 \equiv 0^2$, for $p = 3$ we also have $3 \equiv 0^2$. If $p > 3$, then 2 and 3 may be regarded as elements of the multiplicative group G of a field of residues modulo p. By problem 1772 the group G is cyclic and, by the lemma that was proved above, if 2 and 3 are not squares, then $2 \cdot 3 = 6$ is a square.
The polynomial

$$f(x) = (x-\sqrt{2}-\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x+\sqrt{2}+\sqrt{3}) = x^4-10x^2+1$$

is irreducible over the field of rational numbers since the linear factors and their products taken two at a time are not polynomials with rational coefficients.
Let Z_p be a field of residues with respect to the prime modulus p. From what has been proved there exists an element $a \in Z_p$ for which $a^2 = 2$ or $a^2 = 3$ or $a^2 = 6$. If $a^2 = 2$, then $x^4 - 10x^2 + 1 = (x^2+2ax-1)(x^2-2ax-1)$; if $a^2 = 3$, then $x^4 - 10x^2 + 1 =$

$= (x^2 + 2ax + 1)(x^2 - 2ax + 1)$; if $a^2 = 6$, then $x^4 - 10x^2 + 1 = (x^2 - 5 + 2a)(x^2 - 5 - 2a)$.

1774. *Hint.* Show that if $a = ae$, then e is the unit element.

1775. (a) $a = p$ in the ring of residues modulo p^2; (b) $a = p$ in the ring of residues modulo p^n. Here, p is any integer greater than 1.

1779. *Hint.* For the number $z = a + b\sqrt{-3}$ introduce the norm $N(z) = z \cdot z = a^2 + 3b^2$. Prove that $N(z_1 \cdot z_2) = N(z_1) N(z_2)$; for the given $M > 0$ there only exists a finite set of numbers z with $N(z) < M$; the only unit divisors are ± 1; the divisor z with least norm exceeding 1 is prime.

1780. *Hint.* By transformation of the unknown, carry the irreversible element into a polynomial of degree exceeding two.

1781. (a) An ideal, (b) a subring, (c) an ideal, (d) it is not a subgroup of the additive group, (e) a subring, (f) a subgroup of the additive group, (g) an ideal, (h) a subring, (i) an ideal, (j) an ideal, (k) it is not a subgroup of the additive group.

1785. *Hint.* Show that any ideal I is generated by its nonzero element a that is smallest in the following meaning: (a) in absolute value; (b) in degree; (c) with respect to modulus. In each case make use of the existence of division (with a remainder) by the element $b \neq 0$; the remainder is either equal to zero or is less than the divisor in the sense indicated above.

1791. If $ne \neq 0$ for any $n \neq 0$ in Z, then φ is an isomorphism and $\varphi(Z)$ is isomorphic to Z. If $ne = 0$ for some $n \neq 0$ taken from Z, and n_0 is the smallest positive number for some $n_0 e = 0$, then $\varphi(Z)$ is isomorphic to the ring of residues modulo n_0.

1792. (b) Four cosets consisting of the numbers $a + bi$ with the following properties: (1) a and b are even, (2) a is even and b is odd, (3) a is odd and b is even, (4) a and b are odd; (c) the coset B containing $1 + i$ is a zero divisor, and $B^2 = 0$.

1799. The number of elements is equal to p^n.

1800. 1. R is a ring with zero divisors that is regarded as a module over itself, λ and a are zero divisors for which $\lambda a = 0$. 2. $G = \{a\}$ is a cyclic group of order n (written additively) that is regarded as a module over the ring of integers. Then $na = 0$.

1804. *Hint.* Use problem 1647 (c).

1807. (b) Let a and b be two distinct elements of second order of a four-group [see answer to problem 1638 (b)]. If this group is regarded as a unital module over the ring of integers (under ordinary multiplication of the elements of the group by numbers), then $O(a)$ and $O(b)$ coincide with the set of even numbers, but $\{a\} \neq \{b\}$.

1808. *Hint.* Prove that the set I of all $\lambda \in R$, for which $\lambda a \in A$, is an ideal of the ring R.

1809. The set A of all elements with a finite number of nonzero components.

1810. (b) Example 1. The elements 2 and 3 are periodic in the ring Z_6 of residues modulo 6, as in a module over itself, and their sum 5 is not a periodic element.

Example 2. Let R be a ring of pairs (x, y), where x and y are integers, and addition and multiplication of pairs are carried out componentwise (problem 1734). The elements $a = (1, 0)$ and $b = (0, 1)$ are zero divisors. If R is regarded as a module over itself, then a and b are periodic elements since $O(a)$ is a set of all pairs of the form $(0, y)$

and $O(b)$ is the set of all pairs of the form $(x, 0)$. But the element $a + b = (1, 1)$ has for its order the zero element $(0, 0)$.

1811. *Hint.* Set $\alpha = \alpha'\delta$, $\beta = \beta'\delta$ and show that $\alpha'\beta'\delta(a + b) = \theta$ and $\alpha\gamma b = 0$.

1812. *Hint.* Show that M is a direct sum of the nonzero submodules M_i, each of which consists of all elements of M whose orders are generated by the powers of the single element p_i of R.

1813. *Hint.* Consider the union of the modules M_i.

1819. *Hint.* Show that $b \to A + b$ is a homomorphic mapping of B onto $(A + B)/A$ and use the theorem on homomorphisms for modules.

1820. *Hint.* Use induction on n. For $n = 1$ use problems 1815 and 1818. For $n > 1$ assume that in M there is an infinite increasing chain of distinct submodules $M_1 \subset M_2 \subset \ldots$; set $A = \{x_1\}$, $B = \bigcup_{i=1}^{\infty} M_i$, $M_i' = M_i \cap A$ and show that the chain M_i' becomes stabilized on some $M_k' = A \cap B$. Then apply the isomorphism theorem (see problem 1819) and use the fact that the factor-module M/A has $n - 1$ generators.

1822. *Hint.* When proving (b), consider the expression $(1+1) \times (x + y)$.

1823. (c) The space V is infinite-dimensional.

1825. *Hint.* Prove by induction or, in the linear relation, set $x = 1, 2, 2^2, \ldots, 2^{n-1}$.

1826. *Hint.* In the cases (c), (d), (e) differentiate twice and apply induction.

1827. *Hint.* Use the Vandermonde determinant.

1829. The dimension is equal to $C_{k+n-1}^{n-1} = C_{k+n-1}^{k}$. *Hint.* Take all monomials for the basis and with each monomial of the form $x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ associate the row

$$\underbrace{1 \ldots 1}_{\alpha_1 \text{ times}} x_1 \underbrace{1 \ldots 1}_{\alpha_2 \text{ times}} x_2 \ldots \underbrace{1 \ldots 1}_{\alpha_n \text{ times}} x_n.$$

1830. C_{k+n}^{n}. *Hint.* Set $x_i = \dfrac{y_i}{y_{n+1}}$ and reduce to the preceding problem.

1831. (b) The dimension of L is equal to n; (c) the dimension of L_k is $n - k + 1$; (d) L' is not a subspace.

1832. *Hint.* When proving necessity, obtain the equality $B = CA$, where C is a nonsingular matrix made up of the coefficients in the expressions of the system (2) in terms of (1). When proving sufficiency, adjoin below to matrix A the row of coordinates of the vector b_i and, when computing the rank by the bordering method, show that the rank of the resulting matrix is equal to k.

1836. (d) Suppose, in the plane xOy, $L = Ox$, $M = Oy$, and L' is any straight line that passes through the coordinate origin and is distinct from the axes, φ_1 is a projection on L parallel to M, and φ_2 is a projection on L' parallel to M. Then $\varphi_1\varphi_2 = \varphi_1 \neq \varphi_2 = \varphi_2\varphi_1$. The condition (3) is not fulfilled but $\varphi_1\varphi_2$ and $\varphi_2\varphi_1$ are projections.

Hints. (b) Show that if φ_1 and φ_2 are idempotent, then $\varphi_1 + \varphi_2$ is idempotent if and only if $\varphi_1\varphi_2 + \varphi_2\varphi_1 = 0$. Multiplying this equality on the left and on the right by φ_1, prove that it is equivalent to condition (1). (c) Using (a), reduce (c) to (b). (d) From $\varphi_1\varphi_2 x = x$ derive $\varphi_1 x = \varphi_2 x = x$. Then use the representation $x = \varphi_2 x + (\varepsilon - \varphi_2)\, x$.

1837. *Hint.* Consider $(\varphi(x_1 + x_2), y)$ and $(\varphi(\lambda x), y)$.

1838. $\frac{1}{5}\sqrt{10}$.

1839. If L is the subspace of all vectors, in each of which only a finite number of coordinates are nonzero, then

$$L^* = 0, \quad L \dotplus L^* = L \neq V \quad (L^*)^* = V \neq L.$$

1841. Let A be the transformation matrix in an orthonormal basis. (a) A rotation of the plane through some angle about the coordinate origin if $|A| = +1$; a reflection of the plane in some straight line that passes through the origin if $|A| = -1$; (b) a rotation of space through a certain angle about an axis that passes through the origin if $|A| = +1$; the above-mentioned rotation followed by a reflection of space in the plane passing through the origin and perpendicular to the axis of rotation if $|A| = -1$.

1842. A rotation about the axis defined by the vector $f = (1, 1, 0)$ through the angle $\alpha = 60°$ in the negative direction. *Hint.* We seek the vector f as an eigenvector that belongs to the eigenvalue 1. We find the angle of rotation α from the condition $2\cos\alpha + 1 = a_{11} + a_{22} + a_{33}$ obtained from the invariance of the trace of the transformation matrix φ. To determine the direction of rotation we take a vector that does not lie on the axis of rotation, say e_1, its image φe_1 and the axis vector f, and we seek the sign of the determinant from the coordinates of these three vectors, that is, we seek the orientation of the triplet of vectors e_1, φe_1, f.

1843. (a) The zero transformation; (b) a rotation through the angle $\pi/2$ in the positive or negative direction followed by a multiplication by a nonnegative number; (c) $\varphi x = a \times x$. When $a \neq 0$, the transformation φ reduces to projecting the vector x on the plane perpendicular to the vector a, to the rotation about a through the angle $\pi/2$ in the positive direction, and to multiplication by the length a. *Hint.* Consider the transformation matrix φ in the orthonormal basis $A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$ and set $a = (a_1, a_2, a_3)$, where $a_1 = -a_{23}$, $a_2 = -a_{31} = a_{13}$, $a_3 = -a_{12}$.

1844. *Hint.* When proving sufficiency, consider the scalar product $(\varphi(x + y), x + y)$.

1845. *Hint.* Find the basis $e_1, e_2, \ldots, e_n$ for which

$$l(e_1) = 1, \; l(e_2) = \ldots = l(e_n) = 0.$$

1848. *Hint.* Assume that $l_1(a) \neq 0$, $l_2(b) \neq 0$ and consider the vector $a + b$.

1849. *Hint.* Prove that if $b(x, y) \neq 0$, then

$$\frac{l_1(x)}{l_2(x)} = \frac{l_1(y)}{l_2(y)} = \lambda \neq 0.$$

Consider the product $(l(x) - \lambda l_2(x))\, l_2(x)$ and apply problem 1848.

1850. *Hint.* Use the problems 1846 and 1849.

1851. *Hint.* Use problem 1848.

1852. *Hint.* Set $y = x - \frac{l(x)}{l(a)} \cdot a$.

1853. *Hint.* For nonzero functions, take the vector a not lying in S, set $\lambda = \frac{l_1(a)}{l_2(a)}$, and apply problem 1852 (b).

1854. (a) A one-sheet hyperboloid; (b) a two-sheet hyperboloid. *Hint.* Pass to homogeneous coordinates.

1856. If $e_1, \ldots, e_n$ is a normal basis (in which $f(x)$ can be writen as a quadratic form of normal aspect) and $f(e_i) = 1$ $(i = 1, 2, \ldots, p)$, $f(e_j) = -1$ $(j = p + 1, \ldots, r)$, $f(e_k) = 0$ $(k = r + 1, \ldots, n)$, then for the desired basis we can take, for instance,

$$f_i = e_i + e_{p+1} \ (i = 1, \ldots, p); \quad f_j = -e_p + e_j \ (j = p + 1, \ldots, r);$$

$$f_k = e_k \ (k = r + 1, \ldots, n).$$

Hint. Prove that the vectors

$$f_{ij} = e_i + e_j \ (i = 1, \ldots, p; \ j = p + 1, \ldots, r);$$

$$g_{ij} = e_i - e_j \ (i = 1, \ldots, p; \ j = p + 1, \ldots, r);$$

$h_k = e_k$ $(k = r + 1, \ldots, n)$ are isotropic and the basis $e_1, \ldots, e_n$ is expressed in terms of those vectors.

1857. *Hint.* Use the problems 1306 and 1856.

1858. *Hint.* Consider the case (b) under the condition $p \leqslant q$. Taking the notation $f(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$ for the form of the normal aspect, verify that K contains the subspace L given by the equations

$$x_1 - x_{p+1} = 0, \ \ldots, \ x_p - x_{2p} = 0,$$
$$x_{2p+1} = 0, \ \ldots, \ x_{p+q} = 0 \qquad (1)$$

the dimension L being equal to $n - q$. Then assume that K contains a subspace L' of dimension $s > n - q$ given by the equations

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \quad (i = 1, 2, \ldots, n - s) \qquad (2)$$

adjoin to them the equations

$$x_1 = 0, \ \ldots, \ x_p = 0, \ x_{p+q+1} = 0, \ \ldots, \ x_n = 0 \qquad (3)$$

and obtain a contradiction.

1859. (a) *Solution.* If $f(x)$ is written in the normal form in a suitable basis, then the equation of the surface S can be written thus:

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_n^2 = 1. \qquad (1)$$

If $p = 0$, then there are no real points on S. Here, $\min(p - 1, q) = -1$. The theorem holds if the dimension of the empty set is taken to be equal to -1.

Let $p > 0$. Then there are points on S, for example, $(1, 0, \ldots, 0)$, that is, zero-dimensional manifolds. Let P be a manifold of maximal dimension k appearing in S. P is given by a system of $n - k$ linearly independent equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad (i = 1, 2, \ldots, n-k). \tag{2}$$

This system cannot be homogeneous since the zero solution does not satisfy equation (1). For the sake of simplicity, we assume that the determinant d, of the order $n - k$ and made up of coefficients of the first $n - k$ unknowns, is nonzero. Then the general solution of system (2) may be written down thus:

$$x_i = c_{i,\, n-k+1}x_{n-k+1} + \ldots + c_{in}x_n + c_{i,\, n+1} \quad (i = 1, 2, \ldots, n-k). \tag{3}$$

Consider the $(n + 1)$-dimensional space V_{n+1}. In it we take any basis and assume that V_n is spanned by the first n vectors of the basis.

Consider the homogeneous system of equations

$$\sum_{j=1}^{n} a_{ij}x_j - b_i x_{n+1} = 0 \quad (i = 1, 2, \ldots, n-k). \tag{4}$$

The coefficients are the same as in (2). The general solution is of the form

$$x_i = c_{i,\, n-k+1}x_{n-k+1} + \ldots + c_{in}x_n + c_{i,\, n+1}x_{n+1}$$
$$(i = 1, 2, \ldots, n-k). \tag{5}$$

The coefficients are the same as in (3).

Next consider the cone K specified by the equation

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_n^2 - x_{n+1}^2 = 0. \tag{6}$$

We will prove that the $(k + 1)$-dimensional subspace L given by system (4) lies in the cone K. Any solution of system (4) in which $x_{n+1} = 1$ yields, after x_{n+1} is dropped, the solution of system (2), that is, a vector in $P \subset S$. But such a solution satisfies equation (6) and, hence, lies in K.

If x is any solution of system (4), in which $x_{n+1} = \alpha \neq 0$, then $\frac{1}{\alpha}x \in K$, whence $x \in K$. Let $x = (\alpha_1, \ldots, \alpha_{n-k}, \alpha_{n-k+1}, \ldots, \alpha_n, 0)$ be a solution of system (4) with $x_{n+1} = 0$. There is a solution x^l of system (4) in which the free unknowns have the values

$$x_{n-k+1} = \alpha_{n-k+1}, \ \ldots, \ x_n = \alpha_n, \quad x_{n+1} = \frac{1}{l} \quad (l = 1, 2, \ldots).$$

From the formulas (5) it is clear that $\lim_{l \to \infty} x^l = x$. From what has been proved above, $x^l \in K$. Passing to the limit in (6) after substituting into it the coordinates x^l, as $l \to \infty$, we obtain $x \in K$.

Thus, $L \subset K$. The indices of inertia of K are equal to p, $q+1$. According to problem 1858, $k+1 \leqslant \min(p, q+1)$, whence $k \leqslant \min(p-1, q)$.

Let K' be a cone with the equation

$$\pm x_2^2 \pm \ldots - x_{p+1}^2 - \ldots - x_n^2 = 0, \tag{7}$$

where the signs coincide with the signs of the corresponding terms of equation (1). By problem 1858, the cone K' contains the subspace L' of dimension $k = \min(p-1, q)$ lying in the subspace with the equation $x_1 = 0$. In V_n take the vector $a_0 = (1, 0, \ldots, 0)$. Then S contains the manifold $P' = a_0 + L'$ of dimension $k = \min(p-1, q)$. The proof of assertion (a) is complete.

(b) *Hint.* When $r < n$, reduce (b) to (a) in an r-dimensional space.

1860. (a) 1; (b) 0; (c) 1; (d) 0; (e) 1; (f) 1; (g) $n-1$; (h) $n-2$; (i) 0; (j) 1 if $n > 2$; 0, if $n = 2$, (k) 1; (l) the integral part $\frac{n-1}{2}$ or $\frac{1}{2}n - 1$ if n is even; $\frac{1}{2}(n-1)$ if n is odd.

1861. (b) *Hint.* Find the systems of linear equations that specify the left and right kernels.

1862. The basis L_0' is formed by the vector $(3, -1)$, and the basis L_0'' by the vector $(2, -1)$.

1863. (b) *Hint.* Let $1 < r \leqslant n$. Take a function whose matrix, in a certain basis, has in the upper left-hand corner a nonsingular square block of order r that is neither symmetric nor skew-symmetric, and is elsewhere zero.

1864. *Hint.* Make use of the zero subspace of the function $b(x, y)$.

1865. *Hint.* Show that the coordinates of all vectors in L^* satisfy the matrix equation $BAY = 0$, where A is the matrix of $b(x, y)$ in a certain basis of the space V_n, B is a matrix whose rows involve the coordinates of any basis of the subspace L in the given basis of V_n, and Y is the column of coordinates of the vector $y \in L^*$ in the same basis.

1866. First proof. Since $b(x, y) \not\equiv 0$ but $b(x, x) \equiv 0$, it follows that there are vectors x_1, x_2 such that $b(x_1, x_2) \neq 0$. Multiplying one of these vectors by $\frac{1}{b(x_1, x_2)}$, we obtain the vectors e_1, e_2 for which $b(e_1, e_2) = 1$. The vectors e_1, e_2 are linearly independent, since if $e_2 = \alpha e_1$, then $b(e_1, e_2) = \alpha b(e_1, e_1) = 0$. Let L_1 be a two-dimensional subspace spanned by e_1, e_2 and let L_2 be the set of all $y \in V_n$ such that $b(x, y) = 0$ for any $x \in L_1$. By problem 1865, L_2 is a subspace of dimension $\geqslant n-2$. The intersection of L_1 and L_2 contains only the zero vector, for if $x \in L_1$, then $x = \alpha_1 e_1 + \alpha_2 e_2$. If $x \in L_2$, then $b(e_1, x) = b(e_2, x) = 0$ and $b(e_1, e_1) = b(e_2, e_2) = 0$; $b(e_1, e_2) = -b(e_2, e_1) = 1$, whence $\alpha_1 = 0$, $\alpha_2 = 0$, $x = 0$. By problem 1296, the dimension of L_2 is $n-2$ and V_n is the direct sum of L_1 and L_2. If on L_2 we also have $b(x, y) \not\equiv 0$, then, as above, there are vectors e_3, $e_4 \in L_2$ for which $b(e_3, e_4) = 1$ and so forth. After a finite number of steps, we arrive at the subspace L_{k+1} on which $b(x, y) \equiv 0$. If L_{k+1} is nonzero, then in it we take any basis $e_{2k+1}, \ldots, e_n$. The vectors $e_1, e_2, \ldots, e_n$ form the desired basis.

Second proof. This proof affords a practical method for finding the nonsingular linear transformation of the unknowns, which trans-

formation reduces the given bilinear form to the canonical form given in the problem.

Suppose, in a certain basis, $b(x, y) = \sum_{i, j=1}^{n} a_{ij}x_iy_j$. By changing the numbering of the unknowns we may assume that $a_{12} \neq 0$. Write the form as

$$b(x, y) = x_1(a_{12}y_{12} + \ldots + a_{1n}y_n) - \\ - y_1(a_{12}x_2 + \ldots + a_{1n}x_n) + b_1(x, y)$$

and perform a nonsingular transformation of the unknowns

$$x_1' = x_1,\ x_2' = a_{12}x_2 + \ldots + a_{1n}x_n,\ x_3' = x_3,\ \ldots,\ x_n' = x_n$$

and the same transformation of y_i. This yields

$$b(x, y) = x_1'y_2' - x_2'y_1' + b_2(x, y).$$

If $b_2(x, y)$ does not contain x_2', y_2', then we proceed in a similar fashion. Otherwise $b_2(x, y) = \sum_{i, j=2}^{n} a_{ij}'x_i'y_j'$, where $a_{2k}' \neq 0$ for a certain k, $2 \leqslant k \leqslant n$. Having performed a nonsingular transformation of the unknowns

$$x_1'' = x_1' - a_{23}'x_3' - \ldots - a_{2n}'x_n',\ x_2'' = x_2',\ \ldots,\ x_n'' = x_n'$$

and a similar transformation of y_i', we obtain $b(x, y) = x_1''y_2'' - x_2''y_1'' + b_3(x, y)$, where $b_3(x, y)$ does not contain x_1'', x_2'', y_1'', y_2''. If $b_3(x, y) \neq 0$, we proceed in a similar manner.

1867. $b(x, y) = u_1v_2 - u_2v_1 + u_3v_4 - u_4v_3$; $u_1 = x_1 + 3x_4$; $u_2 = x_2 + 2x_3 - x_4$, $u_3 = x_3$, $u_4 = 6x_4$, and the same expression of v_i in terms of y_i.

1868. $b(x, y) = u_1v_2 - u_2v_1 + u_3v_4 - u_4v_3$; $u_1 = x_1 - 4x_4$, $u_2 = x_2 + 2x_3$, $u_3 = x_3$, $u_4 = -8x_4$, and the same expression of v_i in terms of y_i.

1869. In matrix terms we obtain the following assertion: for a real symmetric matrix A to be orthogonally similar to a matrix in which all the elements of the main diagonal are zero, it is necessary and sufficient that the trace of A be zero.

Hint. When proving sufficiency, use induction on n. For $n > 1$ take any orthonormal basis $f_1, f_2, \ldots, f_n$. If it does not lie on the cone, then prove that there exist vectors f_i, f_j for which $f(f_i) > 0$, $f(f_j) < 0$, and for the first vector of the desired basis take e_1 obtained by normalizing the vector $f_i + \lambda f_j$, where λ is found from the condition $f(f_i + \lambda f_j) = 0$.

1870. *Hint.* Use the property: four distinct points form a parallelogram if and only if their radius vectors satisfy the condition $x_1 + x_3 = x_2 + x_4$.

1875. $x_1 = -1 + 3t_1 - 4t_2$, $x_2 = 2 - t_1 + t_2$,

$x_3 = t_1$, $x_4 = t_2$.

1876. $x_1 = -\frac{1}{2} + \frac{1}{2}t_1 + \frac{1}{2}t_2$,

$x_2 = t_1$, $x_3 = 3 - 4t_2$, $x_4 = 0$, $x_5 = t_2$.

1877. $3x_1 - x_2 - x_3 = 8,\ x_1 - 2x_2 + x_4 = 3,$
$5x_1 - 2x_3 - x_5 = 7.$

1878. $x_1 - 4x_2 + x_3 + 2 = 0,\quad 2x_1 - 3x_2 - x_4 + 7 = 0,$
$3x_1 - 5x_2 - x_5 + 8 = 0.$

1882. Let r and r' be, respectively, the ranks of the matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix};$$

and let r_{ij} and r'_{ij} be, respectively, the ranks of the matrices of the ith and jth rows of the matrices A and A'.

The following five cases are possible for which the indicated values of the ranks are necessary and sufficient:

(1) three planes pass through one point: $r = r' = 3$;

(2) the three planes do not have any points in common but intersect pairwise along straight lines (they form a prism): $r = r_{12} = r_{13} = r_{23} = 2,\ r' = 3$;

(3) two planes are parallel and the third plane intersects them: $r_{12} = 1,\ r = r'_{12} = r_{13} = r_{23} = 2,\ r' = 3$ and two similar cases;

(4) the three planes pass through one straight line: $r = r_{12} = r_{13} = r_{23} = r' = 2$.

(5) the three planes are parallel: $r = 1,\ r'_{12} = r'_{13} = r'_{23} = r' = 2$.

1883. Let r and r' be, respectively, the ranks of the matrices

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}.$$

Four cases are possible:

(1) $r = 3,\ r' = 4$; the straight lines are skew to each other; (2) $r = r' = 3$; the lines intersect; (3) $r = 2,\ r' = 3$; the lines are parallel; (4) $r = r' = 2$; the lines coincide.

1884. Let r and r' be, respectively, the ranks of a simple matrix and an augmented matrix of the combined systems of equations (1) and (2). Five cases are possible:

(1) $r = r' = 4$; the planes intersect in one point; (2) $r = 3,\ r' = 4$; the planes are skew to each other and are parallel to the straight line specified by the three equations of (1), (2) in which the left-hand members are linearly independent; (3) $r = r' = 3$; the planes intersect along a straight line; (4) $r = 2,\ r' = 3$; the planes are parallel; (5) $r = r' = 2$; the planes coincide.

1885. Let r and r' be, respectively, the ranks of the matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 & \dots & a_n & c \\ b_1 & b_2 & \dots & b_n & d \end{pmatrix}.$$

Three cases are possible:

(1) $r = r' = 2$; the hyperplanes intersect along an $(n-2)$-dimensional plane; (2) $r = 1$, $r' = 2$; that is, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} \neq \frac{c}{d}$, the hyperplanes are parallel; (3) $r = r' = 1$, the hyperplanes coincide.

1886. *Hint.* Apply problem 1874 and the appropriate property of subspaces.

1888, 1889, and 1890. *Hint.* Use problem 1887.

1891. If $\pi_1 \parallel \pi_2$, then $\pi_3 = \pi_1$; if $\pi_1 \nparallel \pi_2$, then $\pi_3 = a_1 + (L_1 + L_2)$.

1893. Let the plane π_1 be specified by the system of equations

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= c_1 \\ a_{s1}x_1 + \ldots + a_{sn}x_n &= c_s \end{aligned} \tag{1}$$

and let the plane π_2 be specified by the system of equations

$$\begin{aligned} b_{11}x_1 + \ldots + b_{1n}x_n &= d_1 \\ \ldots\ldots\ldots\ldots\ldots & \\ b_{t1}x_1 + \ldots + b_{tn}x_n &= d \end{aligned} \tag{2}$$

Furthermore, let r_1, r_1' and r_2, r_2' be, respectively, the rank of the matrix made up of the coefficients of the unknowns and the rank of the augmented matrix of the systems (1) and (2); r and r' are, respectively, the rank of the matrix of the coefficients of the unknowns and the rank of the augmented matrix of the combined system consisting of all equations of systems (1) and (2). For the systems (1) and (2) to specify planes, it is necessary and sufficient that each of them be consistent, that is, that $r_1 = r_1'$ and $r_2 = r_2'$. When these conditions are complied with, it is necessary and sufficient for the parallelism of the given planes that $r = \max(r_1, r_2)$, $r' = r + 1$.

1895. (a) The polyhedron P is given by the system of inequalities $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$, $x_4 \geqslant 0$, $x_1 + x_3 \leqslant 1$, $x_1 + x_4 \leqslant 1$, $x_2 + x_3 \leqslant 1$, $x_2 + x_4 \leqslant 1$;

(b) the three-dimensional faces are four quadrangular pyramids: $OABCD$ with vertex D, $OABCE$ with vertex E, $ODEFA$ with vertex A, $ODEFB$ with vertex B, and the four tetrahedrons $ACDF$, $ACEF$, $BCDF$, $BCEF$. *Hint.* Pass a three-dimensional plane through every four points not lying on one two-dimensional plane. If $\sum a_i x_i = b$ is the equation of such a plane and for the coordinates of all the given points we have either $\sum a_i x_i \geqslant b$ or $\sum a_i x_i \leqslant b$, then the corresponding inequality enters into a system of inequalities that specify the polyhedron P. The convex closure of all points lying in the given three-dimensional plane will be a three-dimensional face of the given polyhedron. For example, the points O, A, B, D determine a three-dimensional plane with the equation $x_4 = 0$. For all given points, $x_4 \geqslant 0$. Therefore the inequality $x_4 \geqslant 0$ enters into the desired system. The five given points O, A, B, C, D lie on the three-dimensional plane $x_4 = 0$. Their convex closure is a pyramid which is a three-dimensional face of the polyhedron P. On the contrary, the four points O, A, B, F determine a three-dimensional plane with the equation $x_3 - x_4 = 0$; for the point D we have $x_3 - x_4 > 0$ and for the point E we have $x_3 - x_4 < 0$. Thus, this plane does not lead to the desired

inequality and does not contain the faces of the polyhedron P. In order to reduce the number of quadruplets of points under consideration, take into account that the two quadruplets $OABC$ and $ODEF$ are of an equal status and lie in two-dimensional planes.

1896. (a) The polyhedron P is specified by the system of inequalities $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$, $x_4 \geqslant 0$, $x_1 + x_4 \leqslant 1$, $x_2 + x_4 \leqslant 1$, $x_3 + x_4 \leqslant 1$.

(b) The three-dimensional faces are: the cube $OABCDEFG$ and six quadrangular pyramids with a common vertex H: $OBCFH$, $OACEH$, $OABDH$, $ADEGH$, $BDFGH$, $CEFGH$.

1897. Five vertices $A(1, 1, 1)$, $B(1, 1, -2)$, $C(1, -2, 1)$, $D(-2, 1, 1)$, $E\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$. The polyhedron has six triangular faces ABC, ABD, ACD, BCE, BDE, CDE and constitutes two tetrahedrons $ABCD$ and $BCDE$ with the common base BCD.

1898. (a) A tetrahedron with the vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$;

(b) an octahedron with the vertices $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(0, 1, 0, 1)$, $(0, 0, 1, 1)$;

(c) a triangular prism with bases at the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(1, 0, 0, 1)$, $(0, 1, 0, 1)$, $(0, 0, 1, 1)$;

(d) a square with the vertices $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(0, 1, 0, 1)$.

1899. (a) 8, (b) 28, (c) 14, (d) 2, (e) (7), (f) 4, (g) 7, (h) 6, (i) 3, (j) 3, (k) 1, (l) 6, (m) 5, and (n) 12. *Hint.* Introduce a system of coordinates and consider the parametric equations of the straight line and the plane in vector form.

1901. (a) $a_l = \varphi(e_l)$ $(l = 1, 2, \ldots, n)$.

1902. $F(\varphi) = \varphi(x)$, $\varphi \in V_n^*$.

1904. $A' = C^*AC$, where C is the transition matrix from the old basis to the new basis, written down in columns.

1905. $A' = D^*AD$, where $D = (C^*)^{-1}$, and C is the transition matrix from the old basis to the new basis written down in columns.

1906. $A' = C^{-1}AC = D^*AC$, where C is the transition matrix from the old basis to the new basis written down in columns, and $D = (C^*)^{-1}$.

1907. (b) $F(x, \varphi) = \varphi(x)$, $x \in V_n$, $\varphi_{aj} \in V_n^*$.

1908. *Hint.* (*b*) Take the contraction $a_{i\alpha}b^{\alpha j}$ of the tensor $a_{ij}b^{kl}$, where b^{kl} is the tensor with components $b^{kl} = a^{kl}$ in one basis. Apply problem 1907.

1913. $a'^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} = a^{\pi(j_1)\pi(j_2) \ldots \pi(j_q)}_{\pi(i_1)\pi(i_2) \ldots \pi(i_p)}$.

1914. $a^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} = (-1)^{s+t}$, where s is the number of inversions in the permutation $i_1, i_2, \ldots, i_n$ and t is the number of inversions in the permutation $j_1, j_2, \ldots, j_n$ if the upper and lower indices are distinct; otherwise the indicated coordinate is zero.

1917. It is an invariant equal to the number 0 in all bases.

1918. *Hint.* (a) Verify this for each of the equivalence rules indicated in the introduction to the section; (b) in order to prove necessity for $x \neq 0$, take the contraction with respect to φ with the property $\varphi(x) \neq 0$. To prove sufficiency, put $0 = 0x$ for the pair $x0'$ in the pair

$00'$; (c) take the contraction with $\varphi' \in V'$, for which $\varphi'(x_i') = 1$, $\varphi'(x_j') = 0$ $(j \neq i)$; (d) make use of (c).

1923. (b) $g_{11} = g_{22} = 1$, $g_{12} = g_{21} = \cos\alpha$; (c) $g^{11} = g^{22} = \dfrac{1}{\sin^2\alpha}$; $g^{12} = g^{21} = -\dfrac{\cos\alpha}{\sin^2\alpha}$; (d) $e^1 = \dfrac{1}{\sin^2\alpha}(e_1 - e_2\cos\alpha) = (1, -\cot\alpha)$, $e^2 = \dfrac{1}{\sin^2\alpha}(-e_1\cos\alpha + e_2) = \left(0, \dfrac{1}{\sin\alpha}\right)$; (e) $(x, y) = x^1y^1 + (x^1y^2 + x^2y^1)\cos\alpha + x^2y^2$, where $x = x^1e_1 + x^2e_2$, $y = y^1e_1 + y^2e_2$; (f) $\varepsilon_{12} = -\varepsilon_{21} = |\sin\alpha|$, $\varepsilon_{11} = \varepsilon_{22} = 0$; (g) $S = \varepsilon_{ij}x^iy^j = |\sin\alpha| \begin{vmatrix} x^1 & x^2 \\ y^1 & y^2 \end{vmatrix}$.

1924. When passing to a new basis with the same orientation, y remains unchanged, but if the orientation is changed, the direction is changed. The vector y is obtained from x by a rotation through the angle $\dfrac{\pi}{2}$ in the negative direction with respect to the orientation of the basis e_1, e_2. *Hint.* When determining the dependence of y on the basis, make use of the invariance of tensor equations. To determine the geometric relationship between x and y, consider an orthonormal basis.

1925. When passing to a new basis with the same orientation, the quantities b_{ij} vary as the components of a twice covariant tensor. When passing to a basis with opposite orientation, the quantities b_{ij} change sign as well.

1926. $a_{ijk} = g_{i\alpha}g_{j\beta}a_k^{ij}$ $(i, j, k = 1, 2, \ldots, n)$.

Hint. Contract both sides of the equality given in the problem with $g_{i\alpha'}g_{j\beta'}$ with respect to i and j, make use of the relation $g_{i\alpha'}g^{i\alpha} = \delta_{\alpha'}^{\alpha}$ and then change the notation i, j to α, β and α', β' to i, j.

1928. $S = \dfrac{3}{2}$, $h = 1$. *Hint.* Seek the area via the formula $S = \dfrac{1}{2}bc\sin A$ or make use of problem 1935.

1929. $Q(-4, 2, 0)$. *Hint.* Write the parametric equations of the straight line PQ in contravariant components.

1930. *Hint.* When proving (b), take the orthonormal basis e_1, e_2, e_3, e_4, where e_1 is directed along u, and e_2, e_3, e_4 lie in the same three-dimensional plane as x, y, z and have the same orientation. Make use of the expression of the oriented volume via formulas (17) and (18) of the introduction to this section.

1931. The invariant equal to the number n in any basis. **1933.** 6.

1934. (b) $G_1 = \begin{pmatrix} 6 & -5 & 2 \\ -5 & 5 & -2 \\ 2 & -2 & 1 \end{pmatrix}$, (c) $\left(\dfrac{2}{\sqrt{11}}, \dfrac{1}{\sqrt{11}}, \dfrac{-1}{\sqrt{11}}\right)$ to within sign.

1935. *Hint.* First method: take the given vectors for the basis; second method: choose an orthonormal basis.

1937. $d = \left|\dfrac{\sin\omega(ax_0 + by_0 + c)}{\sqrt{a^2 + b^2 - 2ab\cos\omega}}\right|$. *Hint.* Use problem 1936.

1938. *Hint.* Pass to an orthonormal basis with the same orientation.

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